$\cdots + x^5 + x^6 + x^7.$

2. Let $\dfrac{1}{1+x} = (1+x)^{-1}$. Then $f(0) = 1$.

$$
\begin{aligned}
f'(x) &= -1!(1+x)^{-2} & f'(0) &= -1, \\
f''(x) &= 2!(1+x)^{-3} & f''(0) &= 2!, \\
f'''(x) &= -3!(1+x)^{-4} & f'''(0) &= -3!, \\
f^{(4)}(x) &= 4!(1+x)^{-5} & f^{(4)}(0) &= 4!, \\
f^{(5)}(x) &= -5!(1+x)^{-6} & f^{(5)}(0) &= -5!, \\
f^{(6)}(x) &= 6!(1+x)^{-7} & f^{(6)}(0) &= 6!, \\
f^{(7)}(x) &= -7!(1+x)^{-8} & f^{(7)}(0) &= -7!, \\
f^{(8)}(x) &= 8!(1+x)^{-9} & f^{(8)}(0) &= 8!.
\end{aligned}
$$

$$
\begin{aligned}
P_4(x) &= 1 - x + x^2 - x^3 + x^4, \\
P_6(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6, \\
P_8(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8.
\end{aligned}
$$

3. Let

4. Let $f(x) = \sqrt[3]{1-x} = (1-x)^{1/3}$. Then $f(0) = 1$, and

$$\begin{aligned} f'(x) &= -\tfrac{1}{3}(1-x)^{-2/3} & f'(0) &= -\tfrac{1}{3}, \\ f''(x) &= -\tfrac{2}{3^2}(1-x)^{-5/3} & f''(0) &= -\tfrac{2}{3^2}, \\ f'''(x) &= -\tfrac{10}{3^3}(1-x)^{-8/3} & f'''(0) &= -\tfrac{10}{3^3}, \\ f^{(4)}(x) &= -\tfrac{80}{3^4}(1-x)^{-11/3} & f^{(4)}(0) &= -\tfrac{80}{3^4}. \end{aligned}$$

Then,

$$\begin{aligned} P_2(x) &= 1 - \frac{1}{3}x - \frac{1}{2!}\frac{2}{3^2}x^2 = 1 - \frac{1}{3}x - \frac{1}{9}x^2, \\ P_3(x) &= P_2(x) - \frac{1}{3!}\left(\frac{10}{3^3}\right)x^3 = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3, \\ P_4(x) &= P_3(x) - \frac{1}{4!}\frac{80}{3^4}x^4 = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \frac{10}{243}x^4. \end{aligned}$$

12. Let $f(x) = e^x$. Since $f^{(k)}(x) = e^x = f(x)$ for all $k \geq 1$, the Taylor polynomial of degree 4 for $f(x) = e^x$ about $x =$ is

$$\begin{aligned} P_4(x) &= e^1 + e^1(x-1) + \frac{e^1}{2!}(x-1)^2 + \frac{e^1}{3!}(x-1)^3 + \frac{e^1}{4!}(x-1)^4 \\ &= e\left[1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right]. \end{aligned}$$

26.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)}{x^2}=\lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{4!}\right)=\frac{1}{2}.$$

27. For $f(h)=e^h$, $P_4(h)=1+h+\dfrac{h^2}{2}+\dfrac{h^3}{3!}+\dfrac{h^4}{4!}$. So

(a)

$$\begin{aligned}\lim_{h\to 0}\frac{e^h-1-h}{h^2}&=\lim_{h\to 0}\frac{P_4(h)-1-h}{h^2}\\&=\lim_{h\to 0}\frac{\frac{h^2}{2}+\frac{h^3}{3!}+\frac{h^4}{4!}}{h^2}\\&=\lim_{h\to 0}\left(\frac{1}{2}+\frac{h}{3!}+\frac{h^2}{4!}\right)\\&=\frac{1}{2}.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{h\to 0}\frac{e^h-1-h-\frac{h^2}{2}}{h^3}&=\lim_{h\to 0}\frac{P_4(h)-1-h-\frac{h^2}{2}}{h^3}\\&=\lim_{h\to 0}\frac{\frac{h^3}{3!}+\frac{h^4}{4!}}{h^3}=\lim_{h\to 0}\left(\frac{1}{3!}+\frac{h}{4!}\right)\\&=\frac{1}{3!}=\frac{1}{6}.\end{aligned}$$

Using Taylor polynomials of higher degree would not have changed the results since the terms with higher powers of h all go to zero as $h\to 0$.

28. (a) We use the Taylor polynomial of degree two for f and h about $x=2$.

$$\begin{aligned}f(x)&\approx f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2=\frac{3}{2}(x-2)^2\\h(x)&\approx h(2)+h'(2)(x-2)+\frac{h''(2)}{2!}(x-2)^2=\frac{7}{2}(x-2)^2\end{aligned}$$

Thus, using the fact that near $x=2$ we can approximate a function by Taylor polynomials

$$\lim_{x\to 2}\frac{f(x)}{h(x)}=\lim_{x\to 2}\frac{\frac{3}{2}(x-2)^2}{\frac{7}{2}(x-2)^2}=\frac{3}{7}.$$

(b) We use the Taylor polynomial of degree two for f and g about $x=2$.

$$\begin{aligned}f(x)&\approx f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2=\frac{3}{2}(x-2)^2\\g(x)&\approx g(2)+g'(2)(x-2)+\frac{g''(2)}{2!}(x-2)^2=22(x-2)+\frac{5}{2}(x-2)^2.\end{aligned}$$

Thus,

$$\lim_{x\to 2}\frac{f(x)}{g(x)}=\lim_{x\to 2}\left(\frac{\frac{3}{2}(x-2)^2}{22(x-2)+5(x-2)^2}\right)=\lim_{x\to 2}\left(\frac{\frac{3}{2}(x-2)}{22+5(x-2)}\right)=\frac{0}{22}=0.$$

29. (a) Since the coefficient of the x-term of each f is 1, we know $f_1'(0)=f_2'(0)=f_3'(0)=1$. Thus, each of the fs slopes upward near 0, and are in the second figure.

The coefficient of the x-term in g_1 and in g_2 is 1, so $g_1'(0)=g_2'(0)=1$. For g_3 however, $g_3'(0)=-1$. Thus, g_1 and g_2 slope up near 0, but g_3 slopes down. The gs are in the first figure.

(b) Since $g_1(0)=g_2(0)=g_3(0)=1$, the point A is $(0,1)$.
Since $f_1(0)=f_2(0)=f_3(0)=2$, the point B is $(0,2)$.