Second Practice Exam Solutions

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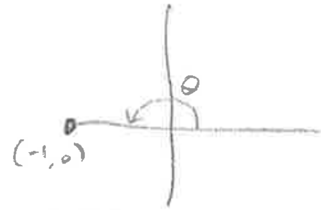
1. Convert to polar.

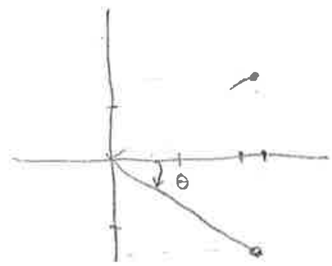
a) $(-1,0)$:



$r = \sqrt{(-1)^2+0^2} = \underline{1}$

$\theta = \pi$ so $(1,\pi)$ in polar

b) $(\sqrt{6}, -\sqrt{2})$



$r = \sqrt{(\sqrt{6})^2+(-\sqrt{2})^2} = \sqrt{8}$

$\theta = \arctan\left(\frac{-\sqrt{2}}{\sqrt{6}}\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right)$

$= -\arctan\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

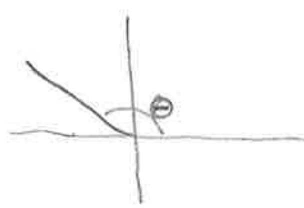
$\tan 30° = 1/\sqrt{3}$

$30° = \frac{\pi}{6}$

so $\left(\sqrt{8}, -\frac{\pi}{6}\right)$ in polar or $\left(\sqrt{8}, \frac{11\pi}{6}\right)$

c) $(-\sqrt{3}, 1)$

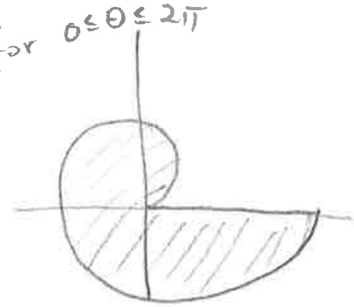
$r = \sqrt{(-\sqrt{3})^2+1^2} = 2$



$\theta = \pi + \arctan\left(-\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

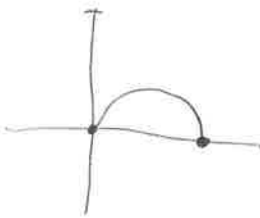
$\left(2, \frac{5\pi}{6}\right)$ in Polar

2) Find the area of $r=\theta$ for $0 \le \theta \le 2\pi$



$$A = \int_0^{2\pi} \frac{r^2}{2}\, d\theta = \int_0^{2\pi} \frac{\theta^2}{2}\, d\theta = \frac{\theta^3}{6}\Bigg|_0^{2\pi} = \frac{(2\pi)^3}{6} = \boxed{\frac{4}{3}\pi^3}$$

3) Find the arclength of $r = 1-\sin\theta$ for $0 \le \theta \le \pi/2$



$$f(\theta) = 1-\sin\theta$$
$$f'(\theta) = -\cos\theta$$

$$L = \int_0^{\pi/2} \sqrt{f^2(\theta) + (f'(\theta))^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{(1-\sin\theta)^2 + (-\cos\theta)^2}\, d\theta = \int_0^{\pi/2} \sqrt{1 - 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2-2\sin\theta}\, d\theta = \sqrt{2}\int_0^{\pi/2} \sqrt{1-\sin\theta}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\sqrt{1-\sin\theta}\left(\sqrt{1+\sin\theta}\right)}{\sqrt{1+\sin\theta}}\, d\theta = \sqrt{2}\int_0^{\pi/2} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1+\sin\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\, d\theta = 2\sqrt{2}\sqrt{1+\sin\theta}\,\Bigg|_{\theta=0}^{\theta=\pi/2}$$

$$\int \frac{\cos\theta}{\sqrt{1+\sin\theta}}\, d\theta = \int \frac{du}{u^{1/2}} = 2u^{1/2} = 2\sqrt{1+\sin\theta}$$

with $u = 1+\sin\theta$

$$= 2\sqrt{2}\left(\sqrt{2}\right) - 2\sqrt{2}\left(\sqrt{1}\right)$$
$$= \boxed{4 - 2\sqrt{2}}$$

Another way to find $\int \sqrt{1-\sin\theta}\, d\theta$ is the following:

Since $\sqrt{1-x^2}$ would have a trig substitution of $x = \sin u$

then we want $\sin\theta = \sin^2 u$

so the derivative is $\cos\theta\, d\theta = 2\sin u \cos u\, du$

$$d\theta = \frac{2\sin u \cos u\, du}{\cos\theta}$$

Since $\sin^2\theta + \cos^2\theta = 1$ then $\cos\theta = \sqrt{1-\sin^2\theta}$

so $\cos\theta = \sqrt{1-\sin^4 u}$

so $$d\theta = \frac{2\sin u \cos u\, du}{\sqrt{1-\sin^4 u}} = \frac{2\sin u \cos u\, du}{\sqrt{1-\sin^2 u}\,\sqrt{1+\sin^2 u}} = \frac{2\sin u \cos u\, du}{\left(\sqrt{1+\sin^2 u}\right)\cos u}$$

so $d\theta = \dfrac{2\sin u\, du}{\sqrt{1+\sin^2 u}}$. Now $\sqrt{1-\sin\theta} = \sqrt{1-\sin^2 u} = \cos u$

so $$\int \sqrt{1-\sin\theta}\, d\theta = \int \frac{2\sin u \cos u\, du}{\sqrt{1+\sin^2 u}}.$$

Now do a u-sub, let $w = 1+\sin^2 u$

so $dw = 2\sin u \cos u\, du$

so

$$\int \frac{2\sin u \cos u\, du}{\sqrt{1+\sin^2 u}} = \int \frac{dw}{\sqrt{w}} = \frac{w^{1/2}}{1/2} + C = 2\sqrt{w} + C$$

$$= 2\sqrt{1+\sin^2 u} + C$$

$$= 2\sqrt{1+\sin\theta} + C$$

(4) $(0.5)^3 + (0.5)^4 + \dots + (0.5)^{2013}$

$$= (0.5)^3 \left(1 + (0.5) + \dots + (0.5)^{2010}\right) = (0.5)^3 \frac{(0.5)^{2011} - 1}{0.5 - 1}$$

$$= (0.5)^3 \frac{1 - (0.5)^{2011}}{1 - 0.5} = (0.5)^2 \left(1 - (0.5)^{2011}\right) = \boxed{\frac{1}{4} - \frac{1}{2^{2013}}}$$

(5) $1 + 2 + 5 + 6 + 9 + 10 + 13 + 14 + \dots + 2013 + 2014$

$$= (1 + 5 + 9 + \dots + 2013) + (2 + 6 + \dots + 2014)$$

$1 + (n-1)4 = 2013$

$n = \frac{2012}{4} + 1 = 503 + 1 = 504$ so $1 + 5 + \dots + 2013 = (504)\left(\frac{1 + 2013}{2}\right)$

and $2 + 6 + \dots + 2014 = (504)\left(\frac{2 + 2014}{2}\right)$

so the total sum is

$$(504)\left(\frac{1 + 2013 + 2 + 2014}{2}\right) = (2015)(504).$$

Another way of doing it is using Gauss's trick:

$S = 1 + 2 + 5 + 6 + \dots + 2013 + 2014$

$S = 2014 + 2013 + 2010 + 2009 + \dots + 2 + 1$

so

$2S = 2015 + 2015 + \dots + 2015 = (2015)(1006)$

so $S = \frac{(2015)(1006)}{2} = (503)(2015)$

(6) $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$

$= a + ar + ar^2 + ar^3 + \cdots$

with $a = 2$ and $r = -1/3$

so the sum is $\frac{a}{1-r} = \frac{2}{1-(-1/3)} = \frac{2}{4/3} = \frac{6}{4} = \boxed{\frac{3}{2}}$

(7)
a) $\lim_{n\to\infty} \frac{3+4n}{5+7n} = \lim_{n\to\infty} \frac{4}{7} = \frac{4}{7}$

so the sequence converges.

b) $\lim_{n\to\infty} \frac{1}{n} + \ln(n) = \lim_{n\to\infty} \ln(n) = \infty$, so the

sequence diverges.

c) $\sin\left(\frac{\pi}{4} n\right)$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\sin(\pi) = 0$

$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\sin\left(\frac{3\pi}{2}\right) = -1$, $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\sin(2\pi) = 0$

The sequence $s_n = \sin\left(\frac{\pi}{4} n\right)$ is

$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \ldots$

it's periodic with period $= 8 > 1$ so it doesn't converge.

It oscilates between these 5 values $\left(\frac{\sqrt{2}}{2}, 1, 0, -\frac{\sqrt{2}}{2}, -1\right)$.

8 a) $\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$

$a_n = \frac{1}{(n+2)^2}$, $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{\frac{1}{(n+2)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+2)^2} = 1$

By the Limit Comparison Test, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

then $\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$ converges.

b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Note that $\frac{n}{2^n} \leq \frac{1}{n^2}$ whenever $n \geq 10$.

Indeed $\frac{10}{2^{10}} \leq \frac{1}{100}$ because $2^{10} = 1024 > 1000$

and 2^n grows faster than n^3 (if $2^n > n^3$ then $2^{n+1} = 2(2^n) > 2n^3$. $2n^3 = n^3 + n^3 > n^3 + 3n^2 + 3n + 1$ for $n \geq 4$)

so

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{10} \frac{n}{2^n} + \sum_{n=11}^{\infty} \frac{n}{2^n} = C + \sum_{n=11}^{\infty} \frac{n}{2^n} < C + \sum_{n=11}^{\infty} \frac{1}{n^2} < C + 2$$

so $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is bounded, therefore it converges.

(Note, we are showing $\sum \frac{n}{2^n}$ converges by using the comparison test with $\sum \frac{1}{n^2}$)

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$: Since $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = 1$

By LIMIT COMPARISON TEST, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

d) $\sum_{n=1}^{\infty} \frac{n^3-2n^2+n+1}{n^5-2}$

By limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

$\sum_{n=1}^{\infty} \frac{n^3-2n^2+n+1}{n^5-2}$ converges.

e) $\sum_{n=1}^{\infty} 2^{-n} \frac{(n+1)}{(n+2)} < \sum_{n=1}^{\infty} \frac{1}{2^n} < \sum_{n=1}^{\infty} \frac{n}{2^n}$

So by comparison test with $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $\sum_{n=1}^{\infty} 2^{-n} \frac{(n+1)}{(n+2)}$ converges.

9 a) $l_1 = (30000)(0.1)$

$l_2 = (30000 - 30000(0.1))(0.1) = 30000(0.1) - 30000(0.1)^2$

$= 30000(0.1)(1-0.1)$

$l_3 = (30000 - 30000(0.1) - 30000(0.1)(1-0.1))(0.1)$

$= (30000)(0.1)(1 - 2(0.1) + (0.1)^2) = 30000(0.1)(1-0.1)^2$

$\vdots$

$$\boxed{l_n = 30000(0.1)(1-0.1)^{n-1}}$$

b) $m_1 = 500$

$m_2 = 500\,(1+0.2)$

$\vdots$

$m_n = 500\,(1+0.2)^{n-1}$

c) Want to find first n s.t $m_n > l_n$.

Let's see when they would be equal:

$$500\,(1+0.2)^{n-1} = 30000\,(0.1)\,(1-0.1)^{n-1}$$

$$\left(\frac{1.2}{0.9}\right)^{n-1} = (60)(0.1) = 6$$

$$(n-1)\ln\left(\frac{1.2}{0.9}\right) = \ln(6)$$

$$n = \frac{\ln(6)}{\ln\left(\frac{1.2}{0.9}\right)} + 1$$

Answer I would expect in the test since you can't use calculators.

$\boxed{\text{So the first integer beyond } \dfrac{\ln(6)}{\ln\left(\frac{1.2}{0.9}\right)} + 1.}$

Since $\dfrac{\ln(6)}{\ln\left(\frac{1.2}{0.9}\right)} + 1 \approx 7.22826\ldots$ then $n = 8$.

At the eight year the maintenance costs more than the value lost.

10 T / F

a) FALSE. Convergence depends on what happens as $n \to \infty$.

b) True. $s_n > 0$ for all n so it's bounded below. If $s_n < 1000\,000$ for all n then s_n is bounded above and hence s_n is bounded. But s_n is unbounded so there must be an n s.t $s_n > 1000\,000$.

c) False. It could be unbounded below. For example $s_n = -n$ satisfies $s_n < 1000000$ for all n, but s_n is unbounded.

d) False. $\sum_{n=1}^{\infty} 2^{(-1)^n} = \frac{1}{2} + 2 + \frac{1}{2} + 2 + \frac{1}{2} + 2 + \dots = \infty$

e) False Example: $a_n = 1$, $b_n = \frac{1}{n^2}$, $\sum \frac{1}{n^2} \cdot 1$ converges but $\sum 1$ diverges

f) True By the comparison test.

g) True By properties of convergence

h) True $\int_0^{\infty} f(7x)\,dx = \int_0^{\infty} \frac{f(u)\,du}{7}$ by doing the u-sub $u = 7x$

so $\int_0^{\infty} f(7x)\,dx = \frac{1}{7} \int_0^{\infty} f(x)\,dx$.

If $\int_0^{\infty} f(x)\,dx$ converges then $\frac{1}{7} \int_0^{\infty} f(x)\,dx$ converges.