Math 53
Homework 1

due January 30 in class

1. Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers and let $f : [1, \infty) \to \mathbb{C}$. For each real number $x \geq 1$, let

$$A(x) = \sum_{n\leq x} a_n$$

and assume that $f(x)$ has a continuous derivative for $x \geq 1$.

a) Assume N is an integer. Prove

$$\sum_{n\leq N} a_n f(n) = A(1)(f(1)-f(2))+\ldots+A(N-1)(f(N-1)-f(N))+A(N)f(N).$$

b) Prove that $f(i+1) - f(i) = \int_i^{i+1} f^{'}(t)\, dt$ and that $A(t)$ is constant on the interval $[i, i+1)$ for i an integer.

c) Using a) and b) prove

$$\sum_{n\leq N} a_n f(n) = A(N)f(N) - \int_1^N A(t)f^{'}(t)\, dt.$$

d) Use a), b) and c) to prove Abel's summation formula, i.e., for any **real** $x \geq 1$

$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f^{'}(t)\, dt.$$

(Note: c) is the proof for x a positive integer, now extend it to any real $x \geq 1$.)

2. Show that the following two representations of the Euler constant γ are actually the same:

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right) \qquad \text{and} \qquad 1-\int_{1}^{\infty}\frac{t-\lfloor t\rfloor}{t^2}dt.$$

3. Let $f:\mathbb{N}\to\mathbb{C}$ be a function for which there exists a positive constant A such that $\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}f(n)=A$. Prove that

$$\sum_{n\le x}f(n)\log n = A(1+o(1))x\log x \qquad \text{(as } x\to\infty).$$

4. A subset A of $\mathbb{N}$ is said to have asymptotic density δ if there exists $0\le\delta\le 1$ such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\le N\\ n\in A}}1=\delta.$$

   a) Let k be a positive integer. Suppose A is the set of positive multiples of k. Show that A has asymptotic density $\frac{1}{k}$.
   b) Let a and b be positive integers. Let $A=\{n\in\mathbb{N}\mid n\equiv b \pmod{a}\}$. Show that A has asymptotic density $\frac{1}{a}$.
   c) Assuming the Prime Number Theorem, i.e., that $\pi(x)\sim\frac{x}{\log x}$, show that the set of prime numbers has asymptotic density 0.
   d) Let A be the set of integers who have first decimal digit equal to 1 (for example, $123\in A$ but $201\notin A$). Show that A does not have an asymptotic density.