Math 53

Homework 4

due February 20 in class

For next week, you have to pick one of these to present in class. The proofs of all of them are in pages 99 and 100 of the textbook. All six exercises combine to prove Theorem 3.17 in the textbook. You need not write up any of the solutions to turn in, you're only homework is to present in class.

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \gamma = \lim_{n\to\infty}\left(\sum_{k\le n}\frac{1}{k} - \log n\right), \quad F(s) = \log\zeta(s) - Z(s),$$

$$Z(s) = \sum_p \frac{1}{p^s}, \quad B_1 = \lim_{n\to\infty}\left(\sum_{p\le n}\frac{1}{p} - \log\log n\right) \qquad B_2 = \sum_p\sum_{k\ge 2}\frac{1}{kp^k},$$

$$H(x) = \sum_{n\le x}\frac{1}{n}, \qquad P(x) = \sum_{p\le x}\frac{1}{p}$$

1. Explain why

$$\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

   and prove

$$F(s) = \sum_{\substack{p\\ k\ge 2}}\frac{1}{kp^{ks}}.$$

   Now show that $F(s) \to B_2$ as $s \to 1$ from the right.

2. Prove

$$\log\zeta(s) = \log\left(\frac{1}{s-1}\right) + O(s-1),$$

   and prove that $\log\left(1-e^{-(s-1)}\right) = \log(s-1) + O(s-1)$ and use this to show

$$\log\zeta(s) = \sum_{n=1}^{\infty}\frac{e^{-(s-1)n}}{n} + O(s-1).$$

3. Prove

$$\log\zeta(s) = (s-1)\int_0^\infty H(t)e^{-(s-1)t}\,dt + O(s-1),$$

and prove

$$Z(s) = (s-1)\int_1^\infty \frac{P(t)}{t^s}\,dt = (s-1)\int_0^\infty P(t)e^{-(s-1)t}\,dt.$$

4. Prove

$$F(s) = (s-1)\int_0^\infty e^{-(s-1)t}\left(\gamma - B_1 + O\left(\frac{1}{t+1}\right)\right)\,dt + O(s-1),$$

and prove

$$(s-1)\int_0^\infty \frac{e^{-(s-1)t}}{t+1}\,dt \le (s-1)\log\frac{s}{s-1} + \frac{s-1}{s}e^{-1}.$$

5. Prove that $F(s) \to \gamma - B_1$ as $s \to 1$ from the right. Conclude that $B_1 + B_2 = \gamma$ and hence

$$\prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{1}{e^\gamma \log x} + O\left(\frac{1}{\log^2 x}\right).$$