Newton’s Binomial

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In class I mentioned Newton’s Binomial theorem, i.e., for $n$ a nonnegative integer and $x, y \in \mathbb{R}$:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k.$$ 

Note that in the formula I point out the symmetry in the exponents of $x$ and $y$ and I also include the fact that $\binom{n}{k} = 0$ for all $k > n$, allowing us to write the sum to infinity.

If you let $y = 1$ you get another version of the Binomial Theorem, namely:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$ 

One reason we let the sum go to infinity is that this way we give ourselves the flexibility to allow $n$ to be a real number and not just an integer. To be able to make sense of $\binom{n}{k}$ when $n$ is a real number we use the fact that

$$\binom{n}{k} = \frac{n(n-1)(n-2)(\cdots)(n-k+1)}{k!},$$

for integers $n, k$ where $n \geq k$ (in fact $k$ can be greater than $n$ and this formula still works, because in that case the binomial coefficient is 0). If we expand this definition of $\binom{n}{k}$ to real numbers we get the more generalized binomial theorem which states that if $n \in \mathbb{R}$, then

$$(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$ 

Let’s look at an application of this formula with the following example:

Let $n = 1/2$ and $x = 1$, then we have

$$(1 + 1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} 1^k = \sum_{k=0}^{\infty} \binom{1/2}{k},$$

hence $\sqrt{2}$ is

$$\begin{align*}
\binom{1/2}{0} + \binom{1/2}{1} + \binom{1/2}{2} + \binom{1/2}{3} + \cdots \\
= 1 + \frac{1}{2} + \frac{1/2(1/2 - 1)}{2!} + \frac{(1/2)(1/2 - 1)(1/2 - 2)}{3!} + \cdots \\
= 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} + \cdots
\end{align*}$$
This gives us a way to approximate $\sqrt{2}$. We can truncate the sum at any point and get better and better approximations. Let $s_n$ be

$$s_n = \sum_{k=0}^{n} \binom{1/2}{k},$$

then

$$s_0 = 1,$$

$$s_1 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5,$$

$$s_2 = 1 + \frac{1}{2} - \frac{1}{8} = \frac{11}{8} = 1.375,$$

$$s_3 = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} = 1.4375,$$

$$s_5 = 1.42578125,$$

$$s_{10} = 1.40993\ldots,$$

$$s_{100} = 1.41407\ldots.$$

Note that $\sqrt{2} = 1.41421\ldots$ The more terms one takes the better the approximation to $\sqrt{2}$.

For those of you wondering why the binomial theorem can work so well with non-integers, think about the Taylor series expansion of $(1 + x)^n$. Using the Taylor series expansion when $n = 1/2$ one can see that in fact that $|\sqrt{2} - s_n| < n^{-3/2}$, so the error when $n = 100$ is at most $100^{-3/2} = .001$. 
