Newton's Binomial

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In class I mentioned Newton's Binomial theorem, i.e., for n a nonnegative integer and $x, y \in \mathbb{R}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k.$$

Note that in the formula I point out the symmetry in the exponents of x and y and I also include the fact that $\binom{n}{k} = 0$ for all $k > n$, allowing us to write the sum to infinity.

If you let $y = 1$ you get another version of the Binomial Theorem, namely:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

One reason we let the sum go to infinity is that this way we give ourselves the flexibility to allow n to be a real number and not just an integer. To be able to make sense of $\binom{n}{k}$ when n is a real number we use the fact that

$$\binom{n}{k} = \frac{n(n-1)(n-2)(\cdots)(n-k+1)}{k!},$$

for integers n, k where $n \geq k$ (in fact k can be greater than n and this formula still works, because in that case the binomial coefficient is 0). If we expand this definition of $\binom{n}{k}$ to real numbers we get the more generalized binomial theorem which states that if $n \in \mathbb{R}$, then

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

Let's look at an application of this formula with the following example:

Let $n = 1/2$ and $x = 1$, then we have

$$(1+1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} 1^k = \sum_{k=0}^{\infty} \binom{1/2}{k},$$

hence $\sqrt{2}$ is

$$\begin{array}{lllll}
\binom{1/2}{0} & +\binom{1/2}{1} & +\binom{1/2}{2} & +\binom{1/2}{3} & +\ldots \\
=1 & +\frac{1/2}{1!} & +\frac{(1/2)(1/2-1)}{2!} & +\frac{(1/2)(1/2-1)(1/2-2)}{3!} & +\ldots \\
=1 & +\frac{1}{2} & +\frac{-1}{8} & +\frac{1}{16} & +\ldots.
\end{array}$$

This gives us a way to approximate $\sqrt{2}$. We can truncate the sum at any point and get better and better approximations. Let s_n be

$$s_n = \sum_{k=0}^{n} \binom{1/2}{k},$$

then

$$\begin{aligned}
s_0 &= 1, \\
s_1 &= 1 + \frac{1}{2} = \frac{3}{2} = 1.5, \\
s_2 &= 1 + \frac{1}{2} - \frac{1}{8} = \frac{11}{8} = 1.375, \\
s_3 &= 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} = 1.4375, \\
s_5 &= 1.42578125, \\
s_{10} &= 1.40993\ldots, \\
s_{100} &= 1.41407\ldots.
\end{aligned}$$

Note that $\sqrt{2} = 1.41421\ldots$. The more terms one takes the better the approximation to $\sqrt{2}$.

For those of you wondering why the binomial theorem can work so well with non-integers, think about the Taylor series expansion of $(1+x)^n$. Using the Taylor series expansion when $n = 1/2$ one can see that in fact that $|\sqrt{2} - s_n| < n^{-3/2}$, so the error when $n = 100$ is at most $100^{-3/2} = .001$.