Induction Proof Practice

① Prove $1+3+6+\dots+\frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$

Proof: By induction

Base case: $n=1$ $\quad 1 = \frac{1(2)(3)}{6}$ ✓

Suppose it's true for $n=k$, i.e.

$$1+3+6+\dots+\frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

now let's consider $n=k+1$

$$1+3+6+\dots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+2)}{6}\left(k+3\right) = \frac{(k+1)(k+2)(k+3)}{6}$$

therefore $1+3+6+\dots+\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$ $\boxed{X}$

② Prove $e^n > n$.

Proof: By induction

Base case: $n=1$ $\quad e > 1$ ✓

Suppose $e^k > k$. (Induction Hypothesis)

Goal: Prove $e^{k+1} > k+1$

$$e^{k+1} = e\cdot e^k > ek \overset{?}{>} k+1$$

$ek > k+1$ because ~~$(e-1)k > k \geq 1$ so~~ $(e-1)k \geq 1$

so $ek > k+1$.

Since $ek > k+1$ and $e^{k+1} > ek$
then $e^{k+1} > k+1$, so the proof is complete. $\boxtimes$

(3) Prove by induction that the number of subsets of a set with n elements is 2^n.

Proof: Base cases: $n=0$ a set with 0 elements has only the empty set as a subset, so it has 1 subset. $1 = 2^0$, so the statement is true.

Induction Hypothesis: Suppose that every set with k elements has 2^k subsets.

Goal: Prove that if A has $k+1$ elements, then A has 2^{k+1} subsets.

Suppose $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$.

Now $\{a_1, a_2, \ldots, a_k\}$ is a set with k elements so it has 2^k subsets, each of of these subsets is also a subset of A. Let's call these the "original" subsets.

Now if you add a_{k+1} to all of those subsets you create 2^k new subsets of A. Let's call these the "new" subsets.

Therefore A has at least $2^k + 2^k = 2^{k+1}$ subsets.

Now we need only verify that it can't have more.

Suppose C is a subset of A not already counted.
Consider 2 cases: Case 1: $a_{k+1} \in C$, Case 2: $a_{k+1} \notin C$.

Case 1: If $a_{k+1} \in C$ then $C - \{a_{k+1}\}$ is a subset of A and in fact a subset of $\{a_1, a_2, \ldots, a_k\}$, so $C - \{a_{k+1}\}$ is of the "original subsets", but that would mean that C is a "new" subset.

But then C is accounted for, contradicting that C is a party crasher.

Case 2: If $a_{k+1} \notin C$ then C is a subset of $\{a_1, a_2, \dots, a_k\}$ and hence an "original" subset. Therefore C is accounted for. Contradicting that C is a party crasher.

Therefore A has exactly 2^{k+1} subsets $\boxtimes$

(4) Prove that every positive integer $n > 1$ has a prime divisor.

Proof: Proof by strong induction.

Base case: $n = 2$, 2 is prime so 2 has the prime divisor 2.

Strong Induction Hypothesis: Suppose that every positive integer greater than 1 and less than or equal to k has a prime divisor, i.e. if $1 < a \leq k$, $a \in \mathbb{N}$, then $\exists p$, $p \mid a$, p prime.

Goal: Prove that $k+1$ has a prime divisor.

If $k+1$ is prime then $k+1$ is the prime divisor of $k+1$.
If $k+1$ is not prime, since $k+1 > 1$ then $k+1$ is composite.
Therefore $\exists\, a, b$ s.t. $k+1 = ab$ with $1 < a < k+1$ and $1 < b < k+1$.
Since $1 < a < k+1$ and $a \in \mathbb{N}$ then $1 < a \leq k$ so by the strong induction hypothesis a has a prime divisor p.
So $p \mid a$. Since $p \mid a$ and $a \mid k+1$ then $p \mid k+1$, so $k+1$ has a prime divisor. $\boxtimes$

Note: A similar proof shows that any positive integer $n > 1$ can be factored into prime factors (example: $30 = 2 \cdot 3 \cdot 5$, $12 = 2 \cdot 2 \cdot 3$, $21 = 3 \cdot 7$, $5 = 5$, $16 = 2 \cdot 2 \cdot 2 \cdot 2$)

That the prime factorization is unique (once you order it) is harder to prove.

(5) $1 \times 1000 + 2 \times 999 + 3 \times 998 + \ldots + 999 \times 2 + 1000 \times 1$

Solution: Let's find a pattern for the sum

$$1 \times n + 2(n-1) + 3(n-2) + \ldots + 2(n-1) + 1(n) = f(n)$$

n	$f(n)$			
1	1		$= \binom{3}{3}$	$= \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$
2	$1 \times 2 + 2 \times 1 = 4$	$= 2 \times 2 = 1 \times 4$	$= \binom{4}{3}$	$= \frac{4 \cdot 3 \cdot 2}{3 \cdot 2}$
3	$1 \times 3 + 2 \times 2 + 3 \times 1 = 10$	$= 2 \times 5$	$= \binom{5}{3}$	$= \frac{5 \cdot 4 \cdot 3}{3 \cdot 2}$
4	$1 \times 4 + 2 \times 3 + 3 \times 2 + 4 \times 1 = 20$	$= 4 \times 5$	$= \binom{6}{3}$	$= \frac{6 \cdot 5 \cdot 4}{3 \cdot 2}$
5	$1 \times 5 + 2 \times 4 + 3 \times 3 + 4 \times 2 + 5 \times 1 = 35$	$= 5 \times 7$	$= \binom{7}{3}$	$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2}$
6	$1 \times 6 + 2 \times 5 + 3 \times 4 + 4 \times 3 + 5 \times 2 + 6 \times 1 = 56$	$= 8 \times 7$	$= \binom{8}{3}$	$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2}$

Guess $f(n) = \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$.

Let's prove it by induction. The base is covered in the table.

Suppose $1 \times k + 2(k-1) + \ldots + (k-1)2 + k(1) = \frac{(k+2)(k+1)k}{6}$

Then $1 \times (k+1) + 2(k) + \ldots + (k-1)(3) + (k)2 + (k+1)(1)$

$= 1 \times k + 1 + 2(k-1) + 2 + \ldots + (k-1)(2) + (k-1) + k(1) + k + k+1$

$= 1 \times k + 2(k-1) + 3(k-2) + \ldots + (k-1)(2) + k(1) + ((1+2+\ldots+k) + k+1)$

$= \frac{(k+2)(k+1)k}{6} + \frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$

so $f(k+1) = \frac{(k+1)(k+2)(k+3)}{6}$ so by induction the proof is complete

So $1 \times 1000 + 2 \times 999 + \ldots + 999 \times 2 + 1000 \times 1 = \boxed{\frac{(1000)(1001)(1002)}{6}}$