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Math 230 Midterm #1
February 7, 2014

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

• You may NOT use a calculator.
• Show all of your work.

<table>
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<th>Question</th>
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1. True or False (Just answer true or false, you don’t need to explain your answer).

(a) [2 points] \( T \subseteq A \) if and only if \( T \in 2^A \).
   \[ \text{TRUE}, \text{ by definition.} \]

(b) [2 points] There is no \( x \) such that \( x \subseteq \{x\} \).
   \[ \text{FALSE}, \quad x = \emptyset \]

(c) [2 points] If \( x \) is a real number and \( x^2 < 0 \), then \( x \) is a perfect number.
   \[ \text{TRUE} \quad \text{(vacuous truth, no real number } x \text{ satisfies } x^2 < 0) \]

(d) [2 points] Two right triangles that have hypotenuses of the same length have the same area.
   \[ \text{FALSE} \quad \text{(counterexample in page 6).} \]

(e) [2 points] \( \exists x, \forall y, xy = 0 \).
   \[ \text{TRUE}, \quad (x=0) \]

(f) [2 points] \( \forall x, \exists y, xy = 0 \).
   \[ \text{TRUE}, \quad (y=0) \]

(g) [2 points] \( \mathbb{N} \in 2^\mathbb{Z} \).
   \[ \text{TRUE} \quad \text{because } \mathbb{N} \subseteq \mathbb{Z}. \]

(h) [2 points] \( \{2\} \subseteq \{\{1\}, \{2\}, \{3\}\} \).
   \[ \text{FALSE} \quad (\exists \emptyset \in \emptyset, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}) \]

(i) [2 points] If \( A \) and \( B \) are sets then \( 2^A \subseteq 2^B \).
   \[ \text{FALSE} \quad \text{it's true if } A \subseteq B. \]

(j) [2 points] A negation of the statement “There is a natural number that is prime and even” can be phrased as “All natural numbers that are prime are odd”.
   \[ \text{TRUE} \]
2. For the following pairs of statements $A$, $B$, write $a$ if the statement “If $A$, then $B$” is true, write $b$ if the statement “If $B$, then $A$” is true, write $c$ if the statement ”$A$ if and only if $B$ is true”, and write $d$ if none of the statements are true. You should write all that apply. Note that in the following, $x$ and $y$ are integers.

(a) [5 points] $A$: $xy = 0$. $B$: $x = 0$ and $y = 0$.

(b) [5 points] $A$: Lines $l_1$ and $l_2$ are parallel. $B$: Lines $l_1$ and $l_2$ are perpendicular.

(c) [5 points] $A$: Joe is a grandfather. $B$: Joe is male.

(d) [5 points] $A$: $x < 0$ $B$: $x^3 < 0$. $A$, $b$, $c$.
3. Proofs:

(a) [10 points] Let $x$ be an integer. Prove that $x$ is odd if and only if there is an integer $b$ such that $x = 2b - 1$.

$$(\Rightarrow) \quad \text{Let } x \text{ be odd.}$$

Then $\exists a \in \mathbb{Z}$ s.t. $x = 2a + 1$.

Therefore $x = 2(a+1) - 1$.

Since $a+1 \in \mathbb{Z}$, let $b = a+1$.

Then $x = 2b - 1$, which is what we wanted.

$$(\Leftarrow) \quad \text{Let } x = 2b - 1 \text{ for some integer } b.$$  

Then $x = 2(b-1) + 1 \implies$

$$x = 2a + 1 \quad \text{where } a \in \mathbb{Z} \quad (a = b-1).$$

So $x$ is odd.
(b) [5 points] For real numbers \( a \) and \( b \), prove that if \( 0 < a < b \), then \( a^2 < b^2 \).

Since \( a < b \) and \( a > 0 \) by the properties of products in inequalities

\[(a)(a) < a(b) \implies a^2 < ab.\]

Since \( b > a \) and \( b > 0 \) then \( b^2 > ab \).

So \( a^2 < ab \) and \( ab < b^2 \).

By the transitive property of "<"

we conclude that \( a^2 < b^2 \).

(c) [5 points] Let \( A, B \) and \( C \) be sets satisfying \( A \subseteq B \) and \( B \subseteq C \). Prove that \( A \subseteq C \).

Let \( x \in A \). Since \( A \subseteq B \implies x \in B \).

\( x \in B \) and \( B \subseteq C \) so \( x \in C \).

Therefore \( A \subseteq C \).
4. Find counterexamples to disprove the following statements:

(a) [5 points] If \(a\), \(b\) and \(c\) are positive integers with \(a|(bc)\), then \(a|b\) or \(a|c\).

\[
6 \big| 2 \cdot 3 \quad \text{but} \quad 6 \nmid 2 \quad \text{and} \quad 6 \nmid 3
\]

no \(b = 2, \quad b = 3\) and \(a = 6\) is a counterexample.

(b) [5 points] Two right triangles have the same area if and only if the lengths of their hypotenuses are the same.

\[
\begin{align*}
\sqrt{31} & = \sqrt{19} + \sqrt{12} \\
\sqrt{5} & = \sqrt{5} + \sqrt{5}
\end{align*}
\]

Both have hypotenuse \(\sqrt{19}\) but their areas are \(\sqrt{19} \cdot \frac{3}{2} = \sqrt{58}\) and \(\sqrt{5} \cdot \sqrt{5} = \sqrt{25}\).

\(\sqrt{58} \neq \sqrt{25}\) so not all right triangles with the same hypotenuse have the same area.

(c) [5 points] For real numbers \(a\) and \(b\), if \(a < b\), then \(a^2 < b^2\).

Let \(a = -2\) and \(b = 1\).

\[
\begin{align*}
\quad a^2 & = 4 \quad \text{and} \quad b^2 = 1 \\
\text{so} \quad a^2 & > b^2 \quad \text{while} \quad a < b.
\end{align*}
\]

(d) [5 points] Let \(A\) and \(B\) be sets. Then \((A \cup B) - B = A\).

If \(A = \{1, 2\}\) and \(B = \{2, 3\}\) then \((A \cup B) - B = \{2, 3\} - \{2, 3\} = \{1\}\).

while \(A = \{1, 2\}\)

so \((A \cup B) - B \neq A\).
5. Boolean Algebra

(a) [5 points] Prove or disprove the following Boolean expression identity:

\[(x \land y) \lor (x \land \neg y) = x.\]

Proof with a truth table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x \land y)</th>
<th>(x \land \neg y)</th>
<th>((x \land y) \lor (x \land \neg y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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Since they have the same column they are logically equivalent.

Alternative Proof: Case 1: \(x = T\) then \((x \land y) \lor (x \land \neg y) = (T \land y) \lor (T \land \neg y) = y \lor (\neg y) = T.\)

Case 2: \(x = F\) then \((x \land y) \lor (x \land \neg y) = (F \land y) \lor (F \land \neg y) = F \lor F = F.\)

(b) [5 points] Besides the classic Boolean operations \(\land, \lor, \neg, \rightarrow, \leftarrow\), we have others, an example of one is the “nand” operation denoted by \(\overline{\land}\). We define \(x \overline{\land} y\) to be \(\neg(x \land y)\). Construct a truth table for \(\overline{\land}\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x \land y)</th>
<th>(\neg(x \land y))</th>
<th>(x \overline{\land} y)</th>
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<td>T</td>
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(c) [5 points] Prove or disprove that \( \wedge \) is commutative.

Let's prove \( x \wedge y = y \wedge x \)

\[
\begin{array}{c|c|c|c|c}
 x & y & x \wedge y & y \wedge x \\
\hline
 T & T & T & T \\
 T & F & F & T \\
 F & T & T & T \\
 F & F & T & T \\
\end{array}
\]

They are equal so \( \wedge \) is commutative.

(d) [5 points] Prove or disprove that \( \wedge \) is associative.

Let's prove \( (x \wedge y) \wedge z = x \wedge (y \wedge z) \)

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
 x & y & z & x \wedge y & (x \wedge y) \wedge z & y \wedge z & x \wedge (y \wedge z) \\
\hline
 T & T & T & T & T & T & T \\
 T & T & F & F & F & F & F \\
 T & F & T & T & T & T & T \\
 T & F & F & F & F & F & F \\
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 F & T & F & F & F & F & F \\
 F & F & T & T & T & T & T \\
 F & F & F & T & T & T & T \\
\end{array}
\]

Since are different, it is not associative.
6. In my comic book library I have 15 Daredevil paperbacks, 12 Spider-man paperbacks and 3 Batman paperbacks

(a) [5 points] In how many different ways can these trade paperbacks be arranged on a bookshelf?

\[ 15 + 12 + 3 = 30. \]

\[ 30! \text{ ways (as factorial).} \]

(b) [5 points] In how many different ways can these trade paperbacks be arranged on a bookshelf if all the books of the same character are grouped together?

\[ 3! \times 15! \times 12! \times 3! \]

- Order the character Daredevils
- Order the Spidermans
- Order the Batman
7. Write out the following sets by listing their elements between curly braces.

(a) [5 points] \( \{ x \in \mathbb{N} : x \leq 10 \text{ and } 3|x \} \).

\[ \{ 3, 6, 9 \} \]

(b) [5 points] \( \{ x \in \mathbb{Z} : x^2 = 4 \} \).

\[ \{ -2, 2 \} \]

(c) [5 points] \( \{ x \in \mathbb{Z} : 10|x \text{ and } x|100 \} \).

\[ \{ 10, -10, 20, -20, 50, -50, 100, -100 \} \]

(d) [5 points] \( \{ x : x \subseteq \{ 1, 2, 3, 4, 5 \} \text{ and } |x| \leq 1 \} \).

\[ \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \}, \{ 5 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 1, 4 \}, \{ 1, 5 \}, \{ 2, 3 \}, \{ 2, 4 \}, \{ 2, 5 \}, \{ 3, 4 \}, \{ 3, 5 \}, \{ 4, 5 \}, \{ 1, 2, 3 \}, \{ 1, 2, 4 \}, \{ 1, 2, 5 \}, \{ 1, 3, 4 \}, \{ 1, 3, 5 \}, \{ 1, 4, 5 \}, \{ 2, 3, 4 \}, \{ 2, 3, 5 \}, \{ 2, 4, 5 \}, \{ 3, 4, 5 \}, \{ 1, 2, 3, 4 \}, \{ 1, 2, 3, 5 \}, \{ 1, 2, 4, 5 \}, \{ 1, 3, 4, 5 \}, \{ 2, 3, 4, 5 \}, \{ 1, 2, 3, 4, 5 \} \} \]
8. Let \( A \times B = \{(1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7), (6, 2), (6, 3), (6, 7)\} \).

(a) [5 points] What is \( A \cup B \)?

\[
A = \{1, 2, 6\} \\
B = \{2, 3, 7\}
\]

\[
A \cup B = \{1, 2, 3, 6, 7\}
\]

(b) [5 points] What is \( A \cap B \)?

\[
A \cap B = \{2\}
\]

(c) [5 points] What is \( A - B \)?

\[
A - B = \{1, 6\}
\]

(d) [5 points] What is \( A \Delta B \)?

\[
A \Delta B = (A - B) \cup (B - A) \\
= \{1, 6\} \cup \{3, 7\} \\
= \{1, 3, 6, 7\}
\]