Not a Ring

Enrique Treviño

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Let

$$S = \{a + b\sqrt[3]{2} \,:\, a, b \in \mathbb{Z}\}.$$

Theorem 1. *S is not a ring.*

Proof. For the sake of contradiction, suppose S is a ring. Since $\sqrt[3]{2} \in S$, then $\sqrt[3]{2} \times \sqrt[3]{2} = \sqrt[3]{4} \in S$. Therefore there exist integers a and b such that

$$a + b\sqrt[3]{2} = \sqrt[3]{4}. \tag{1}$$

Let's show that $a \neq 0$ and $b \neq 0$. If $a = 0$, then $b\sqrt[3]{2} = \sqrt[3]{4}$, therefore $b = \sqrt[3]{2}$. But $1 < \sqrt[3]{2} < 2$, so it is not an integer. This contradicts that b is an integer. Therefore $a \neq 0$. If $b = 0$, then $a = \sqrt[3]{4}$, but then $1 < a < 2$, which means a is not an integer. Therefore $b \neq 0$.

In the (1), we'll cube both sides. We now have

$$4 = a^3 + 3a^2b\sqrt[3]{2} + 3ab^2\sqrt[3]{4} + 2b^2. \tag{2}$$

We're going to use that $a + b\sqrt[3]{2} = \sqrt[3]{4}$ in two different ways. In the first manner we replace $\sqrt[3]{2}$ with $a + b\sqrt[3]{2}$ in the right hand side of (2), so we get

$$4 = a^3 + 2b^3 + 3a^2b\sqrt[3]{2} + 3ab^2(a + b\sqrt[3]{2}) = (a^3 + 2b^3 + 3a^2b^2) + \sqrt[3]{2}(3a^2b + 3ab^3). \tag{3}$$

In the second manner we first factor and then replace $a + b\sqrt[3]{2}$ with $\sqrt[3]{2}$:

$$\begin{aligned} a^3 + 3a^2b\sqrt[3]{2} + 3ab^2\sqrt[3]{4} + 2b^2 &= a^3 + 2b^3 + 3ab\sqrt[3]{2}(a + b\sqrt[3]{2}) \\ &= a^3 + 2b^3 + 3ab\sqrt[3]{2}(\sqrt[3]{4}) \\ &= a^3 + 2b^3 + 6ab. \end{aligned}$$

Therefore

$$4 = a^3 + 2b^3 + 6ab \tag{4}$$

From (3) we have that if $3a^2b + 3ab^3 \neq 0$, then

$$\sqrt[3]{2} = \frac{4 - a^3 - 2b^3 - 3a^2b^2}{3a^2b + 3ab^3}$$

Therefore $\sqrt[3]{2} \in \mathbb{Q}$, i.e., $\sqrt[3]{2}$ is rational because it is the ratio of two integers. Suppose that $\sqrt[3]{2}$ is rational, i.e., there exist integers p and q in lowest terms such that $\sqrt[3]{2} = p/q$. Then

$$\begin{aligned} \frac{p}{q} &= \sqrt[3]{2} \\ \frac{p^3}{q^3} &= 2 \\ p^3 &= 2q^3. \end{aligned}$$

Therefore p^3 is even. Since p^3 is even, then p is even. Therefore $p = 2k$ for some integer k. Then

$$2q^3 = p^3 = (2k)^3 = 8k^3.$$

Therefore $q^3 = 4k^3$, which implies q^3 is even. Since q^3 is even, then q is even. Therefore p and q are even. But p/q is supposed to be in lowest terms. Contradiction! This implies that $\sqrt[3]{2}$ is irrational.

That means that $3a^2b + 3ab^3 = 0$. Plugging this into (3) yields

$$4 = a^3 + 2b^3 + 3a^2b^2 \tag{5}$$

Combining (5) with (4) yields

$$\begin{aligned} a^3 + 2b^3 + 6ab &= a^3 + 2b^3 + 3a^2b^2 \\ 6ab &= 3a^2b^2. \end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, then

$$ab = 2. \tag{6}$$

However, we know that $3a^2b + 3ab^3 = 0$, therefore $a + b^2 = 0$. Then $a = -b^2$. Plugging this into (6) we get

$$-b^3 = 2. \tag{7}$$

But the cubes are 0,1,-1, 8, -8, ..., it does not include 2. Therefore we've reached a contradiction.

Therefore S is not a ring. □