Stirling's Formula

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Stirling's formula is a famous asymptotic formula to compute $n!$, the formula states

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

There are several ways to derive this formula, this note is meant to work on getting approximations to the formula without too much work. In this note we will use the following notation conventions for x a real number:

- $\lfloor x \rfloor$ is the integer part of x, i.e., the largest integer less than or equal to x.
- $\{x\}$ is the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$.
- A function $f(x)$ is said to be $O(g(x))$ if there is a constant C such that $f(x) \leq C|g(x)|$ for large enough values of x.
- A function $f(x)$ is said to be $o(g(x))$ if $\lim_{x \to x} \frac{f(x)}{g(x)} = 0$.
- $\log(x)$ is the natural logarithm of x, i.e., if $y = \log x$ then $e^x = y$.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.

Stirling's formula is equivalent to

$$\log(n!) = n \log n - n + (1/2) \log n + (1/2) \log(2\pi) + o(1).$$

In this note we will prove the following two theorems:

Theorem 1. *For $n \in \mathbb{N}$, $\log(n!) = n \log n - n + O(\log n)$.*

Theorem 2. *For $n \in \mathbb{N}$, $\log(n!) = n \log n - n + (1/2) \log n + O(1)$.*

A quick corollary of Theorem 2 is that

$$n! = O\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right).$$

Proof of Theorem 1. Because $\log(ab) = \log a + \log b$ we have

$$\log(n!) = \log 1 + \log 2 + \log 3 + \ldots + \log n = \sum_{k \leq n} \log n \approx \int_1^n \log t \, dt.$$

By using a left Riemann sum on $\int_1^n \log t \, dt$ we have

$$\begin{aligned}(\log 1 + \log 2 + \log 3 + \ldots + \log(n-1)) + \log n &\leq \left(\int_1^n \log t \, dt\right) + \log n \\ &= n \log n - n + 1 + \log n \\ &= n \log n - n + O(\log n).\end{aligned}$$

On the other hand using a right Riemann sum we get

$$\log 2 + \log 3 + \log 4 + \ldots + \log n \geq \int_1^n \log t \, dt = n \log n - n.$$

Therefore $n \log n - n \leq \log(n!) \leq n \log n - n + 1 + \log n$, and hence $\log(n!) = n \log n - n + O(\log n)$.

□

To prove the second theorem we'll use the following useful result known as "partial summation" or "Abel summation":

Proposition 1 (Abel summation). *Let $\{a_n\}$ be a sequence and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Define $A(x)$ to be*

$$A(x) = \sum_{n \leq x} a_n = a_1 + a_2 + a_3 + \ldots + a_{\lfloor x \rfloor}.$$

Partial summation is the following identity:

$$\sum_{a \leq n \leq b} a_n f(n) = A(b)f(b) - A(a)f(a) - \int_a^b A(t)f'(t) \, dt.$$

Proof of Theorem 2. As stated before

$$\log(n!) = \sum_{1 \leq k \leq n} \log k.$$

Using Abel summation with $a_n = 1$ and $f(n) = \log n$ we have

$$\begin{aligned}\sum_{1 \leq k \leq n} \log k &= n \log n - 1 \log 1 - \int_1^n \frac{\lfloor t \rfloor}{t} \, dt \\ &= n \log n - \int_1^n \frac{t - \{t\}}{t} \, dt \\ &= n \log n - n + 1 + \int_1^n \frac{\{t\}}{t} \, dt \\ &= n \log n - n + \int_1^n \frac{\{t\}}{t} \, dt + O(1).\end{aligned}$$

Therefore proving the theorem comes down to proving that

$$\int_1^n \frac{\{t\}}{t} \, dt = \frac{1}{2} \log n + O(1),$$

which boils down to proving

$$\int_1^n \frac{\{t\} - 1/2}{t} \, dt = O(1). \tag{1}$$

We'll finish by proving (1). We'll break the integral into $n-1$ parts, so that we can be able to get a grip on $\{t\}$:

$$\int_1^n \frac{\{t\} - 1/2}{t} \, dt = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{\{t\} - 1/2}{t} \, dt.$$

Now let's perform the change of variable $t \to t - k$ in the inner integral:

$$\begin{aligned}
\int_k^{k+1} \frac{\{t\} - 1/2}{t} \, dt &= \int_0^1 \frac{t - 1/2}{t + k} \, dt \\
&= 1 - (k + 1/2) \int_0^1 \frac{1}{t+k} \, dt \\
&= 1 - (k + 1/2) \log\left(\frac{k+1}{k}\right) \\
&= 1 - (k + 1/2) \log\left(1 + \frac{1}{k}\right).
\end{aligned}$$

We now use the Taylor series for $\log(1+x) = x - x^2/2 + x^3/3 - \dots$ to get:

$$\begin{aligned}
\left(k + \frac{1}{2}\right) \log\left(1 + \frac{1}{k}\right) &= \left(k + \frac{1}{2}\right)\left(\frac{1}{k} - \frac{1}{2k^2} + \dots\right) \\
&= 1 - \frac{1}{2k} + \frac{1}{3k^2} - \dots + \frac{1}{2k} - \frac{1}{4k^2} + \dots \\
&= 1 + \left(\frac{1}{3k^2} - \frac{1}{4k^2}\right) - \left(\frac{1}{4k^3} - \frac{1}{6k^3}\right) + \left(\frac{1}{5k^4} - \frac{1}{8k^4}\right) \dots \\
&= 1 + \frac{1}{12k^2} - \frac{1}{12k^3} + \frac{3}{40k^4} - \dots \\
&= 1 + O\left(\frac{1}{k^2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_1^n \frac{\{t\} - 1/2}{t} \, dt &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{\{t\} - 1/2}{t} \, \mathrm{d}t \\
&= \sum_{k=1}^{n-1} \left(1 - (k + 1/2) \log\left(1 + \frac{1}{k}\right)\right) \\
&= \sum_{k=1}^{n-1} O\left(\frac{1}{k^2}\right) = O(1).
\end{aligned}$$

$\square$