Final Practice Exam Math 230

1. Prove or disprove that the Boolean expressions $x \to \neg y$ and $\neg(x \to y)$ are logically equivalent.
2. Find counterexamples to disprove the following statements:
   (a) An integer x is positive if and only if $x+1$ is positive.
   (b) An integer is a *palindrome* if it reads the same forwards and backwards when expressed in base 10. For example, 1331 is a *palindrome*. All *palindromes* are divisible by 11.
   (c) If a, b and c are positive integers then $a^{(b^c)} = (a^b)^c$.
   (d) Let A, B and C be sets. Then $A - (B - C) = (A - B) - C$.
3. Let a be an integer. Prove that if $a \geq 3$, then $a^2 > 2a + 1$.
4. Let A, B, C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
5. Prove that the following identities are true for all positive integers n:
   (a) $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$.
   (b) $1 + 10 + 10^2 + \ldots + 10^n = \frac{10^{(n+1)} - 1}{9}$.
6. Prove that the following inequalities are true:
   (a) $e^n > n + 7$, for $n \geq 3$.
   (b) $n^2 \geq 6n + 2$, for $n \geq 7$.
7. Prove that $\sqrt{2}$ is irrational.
8. For each of the following relations defined on the set $\{1, 2, 3\}$ determine whether they are reflexive, irreflexive, symmetric, antisymmetric and/or transitive.
   (a) $R = \{(1,1), (2,2), (3,3)\}$.
   (b) $R = \{(1,1), (2,2), (3,3), (1,2)\}$.
   (c) $R = \{(1,1), (2,2), (1,2), (2,1)\}$.
   (d) $R = \{(1,2), (1,3), (2,3), (2,2)\}$.
9. 51 small insects are in a square of 1×1. Prove that at least three insects are inside a circle of radius $1/7$.
10. True or False:
    (a) Two sets have the same cardinality if there exists a bijection from one of them to the other.

(b) The cardinality of $\mathbb{N}$ is the same as the cardinality of $\mathbb{R}$.
(c) The cardinality of $2^{\mathbb{N}}$ is the same as the cardinality of $\mathbb{R}$.
(d) The cardinality of $(0,1)$ is the same as the cardinality of $[0,1]$.
(e) $f(x) = 2x - 1$ is a bijection from $(0,1)$ to $(0,2)$.
(f) If $f : A \to B$ is onto and $g : B \to C$ is onto, then $g \circ f$ is onto.
(g) Suppose $|A| > |B|$, then there is no one-to-one function $f : B \to A$.
(h) Cantor's theorem states that there is no onto function $f : A \to 2^A$.
(i) Suppose $f : A \to B$ is one-to-one and $g : B \to A$ is one-to-one. Then f is onto.
(j) $f(x) = \tan x$ is a bijection from $(-1,1)$ to $\mathbb{R}$.

11. Determine if the following sets are functions and **explain** why or why not:
    (a) $f = \{(x,y) \mid x + y = 0\}$.
    (b) $f = \{(x,y) \mid xy = 0\}$.
    (c) $f = \{(x,y) \mid x \text{ divides } y\}$.

12. Prove or disprove whether the following functions are one-to-one:
    (a) $f = \{(x,y) \mid x + y = 0\}$.
    (b) $f : \mathbb{R} \to \mathbb{R}, f(x) = 7x - 12$.
    (c) $f = \{(1,1),(2,3),(3,2),(4,3)\}$.

13. Let P be the poset on the set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ defined by the "divides" relation, i.e., a is related to b if $a|b$.
    (a) Does P have a maximum? What about a minimum?
    (b) Find all the maximal elements of P.
    (c) Find all the minimal elements of P.
    (d) What is the height of P?
    (e) What is the width of P?
    (f) Is P a linear order? Why or why not?

14. Let A be the set of all finite posets. Prove that poset isomorphism is an equivalence relation on the set A. (Note: The poset $P = (X, \leq)$ is isomorphic to the poset $Q = (Y, \leq')$ if there exists an order-preserving bijection $f : X \to Y$. Recall that f is order-preserving if for any $a, b \in X$, $a \leq b \Leftrightarrow f(a) \leq' f(b)$.)

15. Let $(X, \leq)$ be a totally ordered set (i.e., it is a linear order). Define the relation $\preceq$ on $X \times X$ as follows. If (x_1, y_1) and (x_2, y_2) are elements of $X \times X$, then we have $(x_1, y_1) \preceq (x_2, y_2)$ provided either $(a) x_1 < x_2$ or else $(b) x_1 = x_2$ and $y_1 \leq y_2$. Prove that $(X \times X, \preceq)$ is a linear order.