Homework 3
SOLUTIONS

8.2 There are 26 letters in the alphabet so
$26 \times 26 \times 26 = 26^3$.

8.4 4 settings, 3 options for air stream, 2 option for the AC button, 4 temperature options, 2 option for recirculate button so there are

$4 \times 3 \times 2 \times 4 \times 2$ possibilities.

$(4 \times 3 \times 2 \times 4 \times 2 = 192)$

8.8 $\boxed{8 \times 7 \times 6}$

8.10 a) $\underbrace{26 \times 26 \times 26}_{\text{letters}} \times \underbrace{10 \times 10 \times 10}_{\text{numbers}} = 26^3 \times 10^3$

b) $26 \times 25 \times 24 \times 10 \times 9 \times 8$

Also accept: $(26)_3 \times (10)_3$ or $\frac{26!}{23!} \cdot \frac{10!}{7!}$

8.15 9 choices for first digit 1, 2, 3, ..., 9
9 choices for second digit (anything but whatever the first digit is).
9 choices for third, fourth and fifth digits (same reason as second digit),
so the answer is

$$9 \times 9 \times 9 \times 9 \times 9 = 9^5.$$

9.2 a) $(6+8+5)! = 19!$

b) 3! ways of choosing the order of the languages, 6! of choosing the order of the french books, 8! for Russian and 5! for Spanish, so there are $(3!)(6!)(8!)(5!)$ of arranging the books.

9.5 $\frac{100!}{98!} = (100)(99) = \boxed{9900}$

9.7. Use calculator to check.

9.9. $1 \times 1! = 1$

$1\cdot 1! + 2\cdot 2! = 5$

$1\cdot 1! + 2\cdot 2! + 3\cdot 3! = 23$

$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! = 119$

$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! + 5\cdot 5! = 719$

Conjecture $\sum_{k=1}^{n} k\cdot k! = (n+1)! - 1$

(Proof for those interested:

$$\sum_{k=1}^{n} k\cdot k! = \sum_{k=1}^{n} (k+1-1)\,k! = \sum_{k=1}^{n} (k+1)! - k!$$

$$= \cancel{2!} - 1! + \cancel{3!} - \cancel{2!} + 4! - \cancel{3!} + \dots + \cancel{n!} - (n-1)! + (n+1)! - n!$$

$$= (n+1)! - 1$$)

9.11. $2 \mid 1000!$ and $2 \mid 2$ so $2 \mid 1000! + 2$

Similarly for $2 \le k \le 1000$

$k \mid 1000!$ and $k \mid k$ so $k \mid 1000! + k$

Therefore $1000! + k$ is composite (since $k < 1000! + k$ and $k \mid 1000! + k$ and $k > 1$)

To check $1000! + 1001$ note $7 \mid 1001$ so $7 \mid 1000! + 1001$

and $1000! + 1002$ is even so it's composite.

Hence $1000! + 2, \dots, 1000! + 1001, 1000! + 1002$ are composite.

A nice corollary of 9.11 is that for any n there are at least n consecutive composite numbers, indeed

$(n+1)!+2, (n+1)!+3, \dots, (n+1)!+(n+1)$ are n consecutive composite numbers.

10.1 a) $\{3, 6, 9\}$

b) $\{2\}$

c) $\{-2, 2\}$

d) $\emptyset$

e) $\{\emptyset\}$

f) $\{-100, -50, -20, -10, 10, 20, 50, 100\}$

g) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$

10.4 a) $\in$ b) $\subseteq$ c) $\in$ d) $\subseteq$ e) $\subseteq$ f) $\subseteq$ g) $\in$

10.9 Proof:

Let $x \in A$.

Since $A \subseteq B$ then $x \in B$.

Since $B \subseteq C$ then $x \in C$.

Therefore $x \in C$. Hence $A \subseteq C$

Let $x \in C$.

Since $C \subseteq A$, then $x \in A$.

Therefore $x \in A$. Hence $C \subseteq A$.

Therefore $A = C$

10.14 The empty set $\emptyset$.

Indeed $\emptyset \subseteq \{\emptyset\}$.

Homework 4
Solutions

11.1 a) $\forall x \in \mathbb{Z}$, such that x is prime.
b) $\exists n \in \mathbb{Z}$, such that n is neither prime nor composite.
c) $\exists n \in \mathbb{Z}$, such that $n^2 = 2$.
d) $\forall n \in \mathbb{Z}$, $5 \mid n$.
e) $\exists n \in \mathbb{Z}$, $7 \mid n$.
f) $\forall n \in \mathbb{Z}$, $n^2 \geq 0$.
g) $\forall x \in \mathbb{Z}$ $\exists y \in \mathbb{Z}$, $xy = 1$.
h) $\exists x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$, $x/y = 10$.
i) $\exists n \in \mathbb{Z}$, $\forall m \in \mathbb{Z}$, $nm = 0$.
j) $\forall n \in \mathbb{Z}$ $\exists m \in \mathbb{Z}$, $m > n$.
k) $\forall p \in$ People $\exists n \in$ people, p loves n.

11.2 a) $\exists x \in \mathbb{Z}$, x is not prime.
b) $\forall n \in \mathbb{Z}$, n is either prime or composite
c) $\forall n \in \mathbb{Z}$, $n^2 \neq 2$.
d) $\exists n \in \mathbb{Z}$, $5 \nmid n$. (Note $a \nmid b$ if a does not divide b)
e) $\forall n \in \mathbb{Z}$, $7 \nmid n$
f) $\exists n \in \mathbb{Z}$, $n^2 < 0$.
g) $\exists x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$, $xy \neq 1$.
h) $\forall x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$ $x/y \neq 10$.
i) $\forall n \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ $nm \neq 0$.
j) $\exists n \in \mathbb{Z}$, $\forall m \in \mathbb{Z}$, $m \leq n$.
k) $\exists p \in$ people, $\forall n \in$ people, p does not love n.

In English:
a) There exists an integer that is not prime.
b) Every integer is either prime or composite.
c) There is no integer n satisfying $n^2 = 2$.
(or every integer n satisfies $n^2 \neq 2$).
d) There is an integer that is not divisible by 5.
e) There is no integer divisible by 7.
f) There is an integer whose square is negative.
g) There is an integer x for which no matter the choice of integer y, $xy \neq 1$.

h) Every integer x and every integer y satisfy that $x/y \neq 10$.

i) For every integer there is an integer it can be multiplied by to get a nonzero number.

j) There is an integer ~~larger which~~ them every other integer.

k) Someone doesn't love anyone anytime.

11.4 a) F ($x=2,\ y=1 \quad 2+1 \neq 0$)
b) T (let $y=-x \quad x+y=0$)
c) F (suppose x exists then $x+x=0$ so $x=0$. But if $x=0,\ y=1$ shows $0+1\neq 0$)
d) T (for any x, $y=-x$ works)
e) F ($x=y=1$ fails)
f) T (let $y=0$)
g) T (let $x=0$)
h) T (let $x=y=0$).

11.7 a) True. Only $x=2$ satisfies $x \in \mathbb{N}$ and $x^2=4$.
b) False. $x=2$ and $x=-2$ work.
c) False. No integer satisfies $x^2=3$.
d) True. Only $x=0$ satisfies the condition.
e) True. Only $x=1$ satisfies the condition.

12.1 $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$

a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
b) $A \cap B = \{4, 5\}$
c) $A \setminus B = \{1, 2, 3\}$
d) $B \setminus A = \{6, 7\}$
e) $A \Delta B = \{1, 2, 3, 6, 7\}$
f) $A \times B = \{(1,4), (1,5), (1,6), (1,7),$
$(2,4), (2,5), (2,6), (2,7),$
$(3,4), (3,5), (3,6), (3,7),$
$(4,4), (4,5), (4,6), (4,7),$
$(5,4), (5,4), (5,6), (5,7)\}$
g) $B \times A = \{(4,1), (4,2), (4,3), (4,4), (4,5),$
$(5,1), (5,2), (5,3), (5,4), (5,5),$
$(6,1), (6,2), (6,3), (6,4), (6,5),$
$(7,1), (7,2), (7,3), (7,4), (7,5)\}$

12.5 Things to prove:
1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap (B \cap C) = (A \cap B) \cap C$
5. $A \cup \emptyset = A$
6. $A \cap \emptyset = \emptyset$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let's prove one by one.

1. $x \in A \cup B$ so $x \in A$ or $x \in B$
therefore $x \in B$ or $x \in A$ so $x \in B \cup A$.
Similarly $x \in B \cup A$ implies $x \in A \cup B$.

2. $x \in A \cap B$ so $x \in A$ and $x \in B$
so $x \in B$ and $x \in A$ so $x \in B \cap A$.

By symmetry (and commutativity of and)
$x \in B \cap A$ implies $x \in A \cap B$.

(3) $x \in A \cup (B \cup C)$ so $x \in A$ or $x \in B \cup C$
so $x \in A$ or $x \in B$ or $x \in C$
so ($x \in A$ or $x \in B$) or $x \in C$
so $x \in (A \cup B) \cup C$.
If $x \in (A \cup B) \cup C$ then
$x \in A \cup B$ or $x \in C$
so $x \in A$ or $x \in B$ or $x \in C$
so $x \in A$ or ($x \in B$ or $x \in C$)
so $x \in A \cup (B \cup C)$.

(4) Same argument as above but change $\cup$ for $\cap$ and "or" for "and".

(5)

($\Rightarrow$)
Let $x \in A \cup \emptyset$ $x \in A$ or $x \in \emptyset$.
In particular $x \in A$
($\Leftarrow$) Let $x \in A \cup \emptyset$, then $x \in A$ or $x \in \emptyset$.
Since there is no x in $\emptyset$, $x \in A$.
So $x \in A$. ☒

(6) $A \cap \emptyset = \emptyset$?
Let $x \in A \cap \emptyset$ then $x \in A$ and $x \in \emptyset$ so $x \in \emptyset$.
Let $y \in \emptyset$ then since $y \in \emptyset$ is false, $y \in A \cap \emptyset$
(False implies anything)
So $A \cap \emptyset = \emptyset$.

(7) $x \in A \cup (B \cap C)$.
$x \in A$ or $x \in B \cap C$
so $x \in A$ or ($x \in B$ and $x \in C$).

$\therefore$ cases:

a) $x \in A$, $x \in B$, $x \in C$

b) $x \in A$, $x \in B$, $x \notin C$

c) $x \in A$, $x \notin B$, $x \in C$

d) $x \in A$, $x \notin B$, $x \notin C$

e) $x \notin A$, $x \in B$, $x \in C$

(note if $x \notin A$, x is forced to be in $B \cap C$ so $x \in B$ and $x \in C$)

In case a) $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$.

In case b) $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$.

In case c) $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$

In case d) $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$

In case e) $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$.

In all cases $x \in (A \cup B) \cap (A \cup C)$.

Now let $y \in (A \cup B) \cap (A \cup C)$

so $y \in A \cup B$ and $y \in A \cup C$.

If $y \in A$ then $y \in A \cup (B \cap C)$.

If $y \notin A$ then $y \in B$ (because $y \in A \cup B$).

Also $y \in C$ because $y \in A \cup C$

So if $y \notin A$, $y \in B \cap C$ so $y \in A \cup (B \cap C)$.

So we conclude $A \cup (B \cup C) = (A \cup B) \cap (A \cup C)$.

8) $(A \cap B) \cup (A \cap C) \stackrel{?}{=} A \cap (B \cup C)$.

let $x \in A \cap (B \cup C)$.

$x \in A$ and ($x \in B$ or $x \in C$).

By Thm 7.2

$(x \in A) \wedge ((x \in B) \vee (x \in C)) = ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$

so $x \in A \cap B$ or $x \in A \cap C$.

so $x \in (A \cap B) \cup (A \cap C)$.

Now let $y \in (A \cap B) \cup (A \cap C)$

Then $(y \in A \text{ and } y \in B)$ or $(y \in A \text{ and } y \in C)$

So by Thm 7.2

$$(y \in A) \text{ and } (y \in B \text{ or } y \in C)$$

so $y \in A \cap (B \cup C)$. $\boxtimes$

NOTE: (Note Thm 7.2 could have been used for all parts, I used different techniques on the different parts to illustrate different ways you could have proved it.)

12.9 Let's disprove it.
Let $A = \{1,2,3\}$, $B = \{1,2\}$ and $C = \{4,5\}$

then $|A \cup B \cup C| = |\{1,2,3,4,5\}| = 5$

but $|A| = 3$, $|B| = 2$ and $|C| = 2$
so $|A| + |B| + |C| = 7 \neq 5$.

12.12 Let's disprove it:
$A = \{1\} = B$ is a counterexample because
$A \Delta B = \emptyset$ so $|A \Delta B| = 0$
$|A| + |B| - |A \cap B| = 1 + 1 - 1 = 1 \neq 0$.

12.21 a) $A = \{1\}$, $C = \{1\}$, $B = \{2\}$

$A - (B - C) = \{1\} - (\{2\} - \{1\}) = \{1\} - \{2\} = \{1\}$.
$(A - B) - C = (\{1\} - \{2\}) - \{1\} = \{1\} - \{1\} = \emptyset$.

So they are different

b) $(A-B)-C = (A-C)-B$

Let's prove it:

Let $x \in (A-B)-C$

so $x \in A \setminus B$ and $x \notin C$

so $x \in A$ and $x \notin B$ and $x \notin C$

so $(x \in A$ and $x \notin C)$ and $x \notin B$

so $x \in A \setminus C$ and $x \notin B$

so $x \in (A \setminus C) \setminus B$.

Now let $y \in (A \setminus C) \setminus B$.

so $y \in A \setminus C$ and $y \notin B$

so $y \in A$ and $y \notin C$ and $y \notin B$.

so $(y \in A$ and $y \notin B)$ and $y \notin C$

so $y \in A \setminus B$ and $y \notin C$

so $y \in (A \setminus B) \setminus C$ $\boxtimes$

c) Let $A = C = \{1\}$, $B = \{1, 2\}$.

$A \cup B = \{1, 2\}$ $\quad A \cup B - C = \{2\}$.

$(A-C) \cap (B-C) = \emptyset \cap (B-C) = \emptyset \neq \{2\}$

So they are not equal.

d) Let $B = \{1, 2\}$ and $C = \{3\}$

then $A = \{1, 2\}$ and $A \cup C = \{1, 2, 3\} \neq B$.

So again, not equal.

e) Counter example: $A = \{1, 2\}$, $C = \{1\} \Rightarrow B = A \cup C = \{1, 2\}$.

$B \setminus C = \{2\} \neq \{1, 2\} = A$.

f) $A = \{1\}$, $B = \{2\}$

$$|A - B| = |\{1\}| = 1$$

but $|A| - |B| = 1 - 1 = 0$

So it is a counterexample.

g) $A = \{1, 2\}$, $B = \{2, 3\}$

$(A - B) \cup B = \{1\} \cup \{2, 3\} = \{1, 2, 3\} \neq A$.

So it is false.

h) $A = \{1, 2\}$, $B = \{2, 3\}$

$(A \cup B) \setminus B = \{1, 2, 3\} \setminus \{2, 3\} = \{1\} \neq A$.

12.24 ~~We'll~~ We'll use the following theorem repeatedly:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof of 12.24: Let A, B, C be finite sets.

$$|A \cup B \cup C| = |(A \cup B) \cup C|$$

$$= |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$$= |A| + |B| - |A \cap B| + |C| - \left(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \right)$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)|$$

Since $(A \cap C) \cap (B \cap C) = A \cap B \cap C$, we conclude that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \square$$

(12.30) A, B, C are sets.

(a) $(\Rightarrow)$

Let $x \in A \times (B \cup C)$

then $x = (a, b)$ where $a \in A$ and $b \in B \cup C$.

so $b \in B$ or $b \in C$.

Therefore $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

Therefore $(a, b) \in (A \times B) \cup (A \times C)$.

$(\Leftarrow)$

Let $x \in (A \times B) \cup (A \times C)$

Therefore $x = (a, b)$ where $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$.

$(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$
translates in Boolean algebra to

$$\big((a \in A) \wedge (b \in B) \big) \vee \big((a \in A) \wedge (b \in C) \big)$$

so

$$(a \in A) \wedge (b \in B \vee b \in C)$$

so $a \in A$ and $(b \in B$ or $b \in C)$

so $(a, b) \in A \times (B \cup C)$

so $x \in A \times (B \cup C)$ $\boxtimes$

(c) $(\Rightarrow)$

Let $x \in A \times (B \setminus C)$.

Let $x = (a, b)$.

Then $a \in A$ and $b \in B \setminus C$

so $a \in A$ and $(b \in B$ and $b \notin C)$

so $a \in A$ and $b \in B$ and $b \notin C$

so $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$

so $(a, b) \in (A \times B) \setminus (A \times C)$

so $x \in (A \times B) \setminus (A \times C)$.

$(\Leftarrow)$

Let $x \in (A \times B) \setminus (A \times C)$

Let $x = (a, b)$

so $x \in (A \times B)$ and $x \notin (A \times C)$

Since $x \in A \times B$ then $a \in A$ and $b \in B$.

Since $a \in A$ and $(a, b) \notin A \times C$ then $b \notin C$.

Therefore $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$

Therefore $(a \in A)$ and $(b \in B$ and $b \notin C)$

so $(a, b) \in A \times (B \setminus C)$

so $x \in A \times (B \setminus C)$

$\boxtimes$