SOLUTIONS
HW 8

Enrique

24.1 a) 1) Yes, it is a function.

2) domain $= \{1,3\}$

image $= \{2,4\}$

3) It is $\underline{1\text{-}1}$.

4) Inverse is $\{(2,1),(4,3)\}$.

b) 1) Yes

2) domain: $\mathbb{Z}$

image: $\{y \in \mathbb{Z} \mid y = 2x \text{ for } x \in \mathbb{Z}\}$, i.e. even integers.

3) Yes.

4) Inverse:
$\{(2x, x) \mid x \in \mathbb{Z}\}$

c) 1) Yes.

2) domain: $\mathbb{Z}$

image: $\mathbb{Z}$

3) Yes.

4) The same function, i.e. $\{(-x, x) \mid x \in \mathbb{Z}\}$.

d) 1) NO.

Because if $x=0$, there is not a unique y.
$0y = 0$ is true for all y.

e) 1) Yes

2) domain: $\mathbb{Z}$

image: $\{x^2 : x \in \mathbb{Z}\}$

3) No because $(-2)^2 = 2^2$.

f) 1) Yes

2) domain: $\emptyset$

image: $\emptyset$

3) Yes

4) $\emptyset$

g) 1) No because there are 2 y's sometimes

Example $\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 = 1$

and $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$

So for $x = \frac{3}{5}$ there are 2 possible y's
$y = \frac{4}{5}$ and $y = -\frac{4}{5}$.

h) 1) No because if $x=1$ and integer y works, so there's not a unique y.

i) 1) Yes
2) dom: $\mathbb{N}$
image: $\mathbb{N}$
3) Yes.
4) $\{(x,x) \mid x \in \mathbb{Z}\}$. (Itself is the inverse).

j) 1) No because if $x=1$ then y could be 0 or 1
$\binom{1}{0} = \binom{1}{1} = 1$.

24.2 $f: \{1,2,3\} \to \{4,5\}$

$f = \{(1,4), (2,4), (3,4)\}$
$f = \{(1,4), (2,4), (3,5)\}$
$f = \{(1,4), (2,5), (3,4)\}$
$f = \{(1,4), (2,5), (3,5)\}$
$f = \{(1,5), (2,4), (3,4)\}$
$f = \{(1,5), (2,4), (3,5)\}$
$f = \{(1,5), (2,5), (3,4)\}$
$f = \{(1,5), (2,5), (3,5)\}$

None of the functions is 1-1

The onto functions are circled.

(24.5) a) $f(2) = -2$

b) $f(2) = 3$

c) $f(2) = (2+1)^{2+1} = 3^3 = 27$

d) $f(2) = 1$

e) $f(2) = 2! = 2$

(24.6) a) $\mathbb{Z}$ (all integers)

b) $\mathbb{N} \cup \{0\}$ (nonnegative integers)

c) $\mathbb{Z}$ (all integers)

d)

$y = \frac{1}{1+x^2}$ $\quad 1+x^2 = \frac{1}{y}$ $\quad x^2 = \frac{1}{y} - 1$

$x^2 \geq 0$ so $\frac{1}{y} - 1 \geq 0$

so $\frac{1}{y} \geq 1$ so $y \leq 1$.

$\frac{1}{1+x^2} > 0$ for all x so $y > 0$.

Therefore the image is $\{y \in \mathbb{R} \mid 0 < y \leq 1\}$

i.e. the interval $(0, 1]$.

e) $\{y \in \mathbb{R} \mid y \geq 0\} = [0, \infty)$.

f) $[-1, 1]$.

(24.8) a) $(1,6)$ (other possible answers are $(1,7), (2,6), (2,7), (3,5), (3,7)$)

b) $(4,5)$ (other answers are $(4,6)$).

c) $(4,7)$ (no other possible answer).

24.14

a) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$

is 1-1 because if $f(a) = f(b)$

then $2a = 2b$

so $a = b$.

is not onto because there is no $x \in \mathbb{Z}$ s.t $f(x) = 1$.

b) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 10 + x$

is 1-1 because if $f(a) = f(b)$ then

$10 + a = 10 + b$

so $a = b$.

is onto because if $y \in \mathbb{Z}$ $f(y-10) = 10 + (y-10) = y$

and $y - 10 \in \mathbb{Z}$.

c) $f: \mathbb{N} \to \mathbb{N}$, $f(x) = 10 + x$

is 1-1 because if $f(a) = f(b)$ then

$10 + a = 10 + b$

so $a = b$.

is not onto because there is no $x \in \mathbb{N}$ s.t

$f(x) = 1$.

Indeed $f(-9) = 1$ but $-9 \notin \mathbb{N}$.

d) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$

f is not 1-1 because

$f(0) = 0$ and $f(1) = 0$,

f is onto because if $y \in \mathbb{Z}$ then $f(2y) = y$.

and $2y \in \mathbb{Z}$.

e) $f: \mathbb{Q} \to \mathbb{Q}$ $f(x) = x^2$

f is not 1-1 because $f(-1) = f(1)$.

f is not onto because $f(x) \geq 0$ so there is no $x \in \mathbb{Q}$

s.t $f(x) = -1$

24.16 Three things to prove. Let $f: A \to B$
A, B finite sets

1) If f is 1-1 and f is onto $\Rightarrow |A| = |B|$
2) If f is 1-1 and $|A| = |B| \Rightarrow$ f is onto.
3) If f is onto and $|A| = |B| \Rightarrow$ f is 1-1.

Let's prove them in that order

1) Since f is 1-1 $|B| \geq |A|$.
Since f is onto $|A| \geq |B|$.
Therefore $|A| = |B|$

2) Since f is 1-1, $|\mathrm{Im} f| = |A|$

(because for every element in A, there's exactly one element in Im f)

Since $|A| = |B|$ then $|\mathrm{Im} f| = |B|$.
Since $\mathrm{Im} f \subseteq B$ and $|\mathrm{Im} f| = |B|$
then $\mathrm{Im} f = B$.
Therefore f is onto.

3) Since f is onto $\mathrm{Im} f = B$ so $|\mathrm{Im} f| = |B| = |A|$.
Since $|A| = |\mathrm{Im} f|$
then f is 1-1 $\boxtimes$

Homework 9 Solutions
Math 230

24.17: Let $A = \mathbb{N}$. Let

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Then $f : A \to A$ is onto because if $m \in \mathbb{N}$, then $f(2m) = m$ and f is not one-to-one because $f(3) = f(2)$ (both equal 1).

Let $g : A \to A$ be defined by $g(n) = 2n$. Then g is one-to-one because if $g(m) = g(r)$, then $2m = 2r$ so $m = r$. g is not onto because there is no integer n such that $g(n) = 1$. Indeed if $g(n) = 1$, then $2n = 1$, so $n = 1/2$, but $1/2 \notin \mathbb{N}$.

f and g don't contradict Exercise 24.16 because the set $A = \mathbb{N}$ is infinite and the exercise is true for finite sets.

24.20: The answer is

$$\binom{n}{k}.$$

The reason is that if there are k elements of A that map to 1, then we must choose the k elements of A which map to 1 and all other elements of A map to 0. There are $\binom{n}{k}$ ways of choosing which elements of A map to 1. The elements not chosen to map to 1, must map to 0. So each choice of k elements of A represents a unique function $f : A \to \{0, 1\}$ that has exactly k elements $a \in A$ satisfying $f(a) = 1$ (and hence $n - k$ elements b such that $f(b) = 0$).