Homework 7 Solutions
Math 230

November 13, 2015

25.2: We would need 367. There are 366 possible birthdays (including February 29), so we need 367 to guarantee at least two people with the same birthday.

25.6: If two numbers match zeroes, then they satisfy that the zeroes of both numbers are in the same positions among the 9 digits of the numbers. There are 512 configurations of 0's and not 0's (indeed each digit is either a 0 or not a 0 and there are 9 digits, hence $2^9 = 512$ configurations). Since we have 513 numbers, by the Pigeonhole principle at least two of them have the same configuration of 0's. Hence they match zeroes.

25.7: Consider the ones digit of the numbers. Consider the sets $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{0\}, \{5\}$. There are seven integers and six sets. By pigeonhole principle there should be at least two integers that have their ones digit (the remainder when dividing by 10) in one of these sets. Call these numbers a and b. Then, either a and b have the same ones digits, and then $a - b$ is a multiple of 10, or they have a different ones digit, but then we have either 1+9, 2+8, 3+7 or 4+6 and in all cases we get a multiple of 10.

25.9: Break the square into 4 squares of side length $1/2 \times 1/2$ (i.e. draw the lines connecting the midpoints of opposing sides of the square). Since there are 5 points, at least two of them must land in the same $1/2 \times 1/2$ square. The farthest apart two points can be inside the square is if they are in opposing corners, hence

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

apart. This is what we wanted to prove.

25.13: We'll write the proof in a similar way to the proof of the Erdős-Szekeres theorem in the book. Let S be our sequence of 1001 numbers. Let i be the i-th element of the sequence. Let u_i (u for up) be the length of the longest increasing subsequence of S starting at i-th element of the sequence. Let d_i (d for down) be the length of the longest decreasing subsequence of S starting at the i-th element of the sequence. Let c_i (c for constant) be the length of the longest constant subsequence of S starting at the i-th element of S. For each i consider the triple (u_i, d_i, c_i). For the sake of contradiction, suppose that there is no increasing sequence of length 11 (or more), no decreasing sequence of length 11 (or more) and no constant subsequence of length 11 (or more). Then $u_i, d_i, c_i \leq 10$ for all i. But $u_i, d_i, c_i \geq 1$. Therefore there are at most $10^3 = 1000$ possible triples (u_i, d_i, c_i). Therefore, by Pigeonhole

Principle, there are at least two numbers $i < j$ such that $(u_i, d_i, c_i) = (u_j, d_j, c_j)$. Let a be the i-th element of the sequence and b be the j-th element of the sequence. We have three possibilities for a and b.

Case 1: If $a < b$, then $u_i \geq u_j + 1$ because we can incorporate a to the longest increasing subsequence starting at b. Therefore $u_i \neq u_j$, which is a contradiction.

Case 2: If $a > b$, then $d_i \geq d_j + 1$ because we can incorporate a to the longest decreasing subsequence starting at b. Therefore $d_i \neq d_j$. Contradiction.

Case 3: If $a = b$, then $c_i \geq c_j + 1$ because we can incorporate a to the longest constant subsequence starting at b. Therefore $c_i \neq c_j$. Contradiction.

In all three cases we reach a contradiction. Therefore our assumption that there isn't an increasing subsequence of length 11, or a decreasing subsequence of length 11 or a constant subsequence of length 11 is false. Therefore there is such a subsequence.

Remark 1. *You might note I defined u_i in terms of the i-th element of the sequence and not u_x in terms of the element x of the sequence. The reason is that the integers are not necessarily distinct, so if we define u_x and there are multiple x's in the sequence, the definition is ambiguous. By stating that I am defining it in terms of the placement in the sequence, we break the ambiguity.*

26.1:

a) $f \circ g = \{(2,2),(3,2),(4,2)\}$ and $g \circ f = \{(1,1),(2,1),(3,1)\}$. $g \circ f \neq f \circ g$.
b) $f \circ g = \{(2,2),(3,3),(4,4)\}$ and $g \circ f = \{(1,1),(2,2),(3,3)\}$. $g \circ f = f \circ g$.
c) $f \circ g$ is undefined. $g \circ f = \{(1,0),(2,5),(3,3)\}$. $g \circ f \neq f \circ g$.
d) $f \circ g = \{(1,4),(2,4),(3,1),(4,1)\}$ and $g \circ f = \{(1,4),(2,4),(3,4),(4,1)\}$. $g \circ f \neq f \circ g$.
e) $f \circ g = \{(1,4),(2,5),(3,1),(4,2),(5,3)\} = g \circ f$. $g \circ f = f \circ g$.
f)

$$f \circ g(x) = f(x^2+1) = (x^2+1)^2 - 1 = x^4 + 2x^2,$$

and

$$g \circ f(x) = g(x^2-1) = (x^2-1)^2 + 1 = x^4 - 2x^2 + 2.$$

$g \circ f(0) \neq f \circ g(0)$, so $g \circ f \neq f \circ g$.
g)

$$f \circ g(x) = f(x-7) = (x-7) + 3 = x - 4,$$

and

$$g \circ f(x) = g(x+3) = (x+3) - 7 = x - 4.$$

Therefore $g \circ f(x) = f \circ g(x)$.
h)

$$f \circ g(x) = f(2-x) = 1 - (2-x) = x - 1,$$

and

$$g \circ f(x) = g(1-x) = 2 - (1-x) = x + 1.$$

$g \circ f(0) \neq f \circ g(0)$, so $g \circ f \neq f \circ g$.
i) $f \circ g$ is undefined because $g(-1) = 0$ so $f(g(-1))$ is undefined.

$$g \circ f(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1.$$

Since $f \circ g$ is undefined, then $g \circ f \neq f \circ g$.
j) Since $A \neq B$ and $A \subseteq B$, there is an $x \in B$ such that $x \notin A$. For this x, $g(x) = id_B(x) = x$, but $f(x) = id_A(x)$ is undefined. Therefore $f \circ g$ is undefined.

$$g \circ f(x) = g(f(x)) = g(id_A(x)) = g(x) = id_B(x) = x.$$

Since $f \circ g$ is undefined, then $g \circ f \neq f \circ g$.

26.7: Let A and B be sets and $f : A \to B$ and $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. We want to prove that f is invertible, i.e, that f is one-to-one. We also want to prove that $g = f^{-1}$.

Let's start by proving that f is one-to-one. Suppose that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, so $g \circ f(x) = g \circ f(y)$, but $g \circ f = id_A$, so $id_A(x) = id_A(y)$, and therefore $x = y$. Hence f is one-to one, which implies that f is invertible.

Let's now prove that $g = f^{-1}$:

We'll start by proving that f is onto. Let $y \in B$. Since $y \in B$, then $g(y) \in A$. Now $f \circ g(y) = id_B(y) = y$ and $f \circ g(y) = f(g(y))$. So $f(g(y)) = y$, so f is onto.

Since f is one-to-one, f^{-1} exists and its domain is the image of f. Since f is onto, the image of f is B, so $f^{-1} : B \to A$. Therefore the domain of f^{-1} equals the domain of g.

Now we just need to prove that for $y \in B$, $f^{-1}(y) = g(y)$. Since f is onto, there exists an $x \in A$ such that $f(x) = y$. Therefore

$$f^{-1}(y) = f^{-1}(f(x)) = f^{-1} \circ f(x) = id_A(x) = x,$$

and

$$g(y) = g(f(x)) = g \circ f(x) = id_A(x) = x.$$

Therefore $f^{-1}(y) = g(y)$. Therefore $g = f^{-1}$.

26.9: Let A, B, C be sets and $f : A \to B$ and $g : B \to C$.

a) Suppose f and g are one-to-one. Let's prove $g \circ f$ is also one-to-one. Suppose $g \circ f(x) = g \circ f(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one then $f(x) = f(y)$. Since f is one-to-one then $x = y$. Hence $g \circ f$ is one-to-one.

b) Suppose f and g are onto. Let's prove $g \circ f$ is onto. Suppose $c \in C$. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. Therefore $g(f(a)) = c$. Therefore $g \circ f(a) = c$. Therefore $g \circ f$ is onto.

c) Suppose f and g are bijections. Let's prove $g \circ f$ is a bijection. Since f and g are one-to-one, then $g \circ f$ is one-to-one. Since f and g are onto, then $g \circ f$ is onto. Since $g \circ f$ is one-to-one and onto, then $g \circ f$ is a bijection.

26.10: The functions in (e) of exercise 26.1 work, i.e., $A = \{1, 2, 3, 4, 5\}$ with

$$f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\},$$

and

$$g = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}.$$

Note that g is not the inverse of f, that neither is the identity and that

$$f \circ g = \{(1, 4), (2, 5), (3, 1), (4, 2), (5, 3)\} = g \circ f.$$

You might be wondering "what's so special about these two functions?" Both of them are permutations of the set $\{1, 2, 3, 4, 5\}$, one of them is a translation by 1 and the other by 2. So the composition is translating by 3. This kind of construction can be easily generalized to find many more functions with the property of $f \circ g = g \circ f$. An interesting question is whether we can characterize all of the pairs of functions (f, g) with such a property.