Induction Proof Practice

1. Prove that for any positive integer n,
$$1+3+6+\cdots+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}.$$

2. Prove that for any positive integer n,
$$2^n > n.$$

3. Prove by induction that the number of subsets of a set with n elements is 2^n.

4. Prove that every positive integer $n > 1$, has a prime divisor.

5. Evaluate the sum
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{999\cdot 1000}.$$

Solutions

1. *Proof.* For $n=1$, the left side is 1 and the right side is $\frac{1\cdot 2\cdot 3}{6}=1$.

   Suppose the statement is true for $n=k$, namely, suppose that for some $k \geq 1$, we have
$$1+3+\cdots+\frac{k(k+1)}{2}=\frac{k(k+1)(k+2)}{6}.$$

   Now, consider the case $n=k+1$. We have
$$\begin{aligned}1+3+\cdots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2} &= \left(1+3+\cdots+\frac{k(k+1)}{2}\right)+\frac{(k+1)(k+2)}{2}\\ &= \frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}\\ &= \frac{(k+1)(k+2)}{6}(k+3)=\frac{(k+1)(k+2)(k+3)}{6}.\end{aligned}$$

   Therefore, we've shown that when it's true for k, it implies it's true for $k+1$. We've finished the proof by induction.

   $\square$

2. *Proof.* The base case is $n=1$ and we can see that $2^1=2>1$. Therefore it's true for $n=1$.

   Let's assume that it's true for $n=k$, namely, suppose $2^k > k$. We have $2^{k+1}=2\cdot 2^k > 2\cdot k \geq k+1$ whenever $2k \geq k+1$, which is true for $k \geq 1$. Therefore, $2^{k+1} > k+1$ and hence we've proved the general statement by induction.

   $\square$

3. *Proof.* For $n = 0$, we have that the only subset of a set with zero elements is the empty set. Therefore, it has one subset. But $2^0 = 1$, so the statement is true for $n = 0$. For $n = 1$, let $A = \{a\}$ be our set with one element. Then the only subsets are $\emptyset$ and $\{a\}$. Therefore, it has two subsets. Since $2^1 = 2$, we have that the statement to be proved is true for $n = 1$. We have out base case.

   Now, for an integer $k \geq 1$, suppose that the number of subsets of a set with k elements is 2^k. This will be the induction hypothesis.

   Suppose $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ is a subset with $k+1$ elements. Let's consider all the subsets. Let T be the set of subsets of A that contain a_{k+1} and U be the set of subsets that don't contain a_{k+1}. Note that the set of subsets of A is the disjoint union of T and U. We're going to show that $|T| = |U| = 2^k$. First, let's consider U. The subsets of A that don't contain a_{k+1} are the subsets of $\{a_1, a_2, \ldots, a_k\}$. By the induction hypothesis, there are 2^k of these. Now consider the subsets of A that contain a_{k+1}. Once you ignore that term, the rest of the subset must be a subset of $\{a_1, a_2, \ldots, a_k\}$, so by the induction hypothesis there are 2^k of these. Therefore, the number of subsets of A is $|T| + |U| = 2^k + 2^k = 2^{k+1}$, which is what we wanted to prove.

   □

4. *Proof.* For $n = 2$, the prime divisor is 2. Suppose that all numbers $1 < i \leq k$ have a prime factor. We want to show that $k + 1$ also has a prime factor. If $k + 1$ is prime, then it has a prime factor (itself). If $k + 1$ is not prime, then there exist a, b satisfying $1 < a \leq b < k + 1$ such that $k + 1 = ab$. But then $1 < a \leq k$. By the strong induction hypothesis, a has a prime factor p. But then $p|a$ and $a|k+1$, so $p|k+1$. Therefore $k+1$ has a prime factor. Therefore, by strong induction, all integers greater than 1 have a prime factor.

   □

5. Let's find a pattern:

$$\frac{1}{1\cdot 2} = \frac{1}{2}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}$$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}.$$

   It seems the pattern is that the sum up to $\frac{1}{(k-1)k}$ is $\frac{k-1}{k} = 1 - \frac{1}{k}$. This suggests the answer to the question is $\frac{999}{1000}$. Let's prove that the pattern persists by using induction:

   Proof. The base case are the examples listed above. As our induction hypothesis suppose

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k-1)k} = \frac{k-1}{k}.$$

Now, consider the next term:

$$
\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k-1)k}+\frac{1}{k(k+1)} &= \frac{k-1}{k}+\frac{1}{k(k+1)} \\
&= \frac{1}{k(k+1)}\left((k-1)(k+1)+1\right) \\
&= \frac{1}{k(k+1)}(k^2-1+1) \\
&= \frac{k^2}{k(k+1)} \\
&= \frac{k}{k+1}.
\end{aligned}
$$

This completes the proof by induction.

□