Name: ______________________________

Math 230 Midterm #1

September 30, 2013

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	20	
3	10	
4	20	
5	10	
6	10	
7	10	
8	20	
9	10	
Total:	130	

Official Cheat Sheet

1. Let A be a set. Then 2^A is the set of all subsets of A. For example, if $A = \{1,2\}$, then $2^A = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.
2. $|A|$ is the number of elements of A. A useful formula is: $|A \cup B| = |A| + |B| - |A \cap B|$ if A and B are finite sets. Another useful formula is $\left|2^A\right| = 2^{|A|}$ when A is finite.
3. Here are some Boolean algebra properties (which can be translated easily to set properties by equating $\vee$ with $\cup$ and $\wedge$ with $\cap$):
   - $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
   - $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$.
   - $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
4. $\mathbb{Z}$ is the set of integers. $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers.
5. Let A and B be sets. Then
   - $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
   - $A \cap B = \{x | x \in A \text{ and } x \in B\}$,
   - $A - B = \{x | x \in A \text{ and } x \notin B\}$,
   - $A \Delta B = (A - B) \cup (B - A)$.
   - $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
6. Let a and b be integers.
   - a is even if there exists an integer c such that $a = 2c$.
   - a is odd if there exists an integer c such that $a = 2c + 1$.
   - We say $a | b$ (a divides b) if there exists an integer c such that $b = ac$.
   - a is composite if $|a| > 1$ and there exists c such that $1 < c < |a|$ and $c | a$.
   - a is prime if $a > 1$ and a is not composite.
   - a is perfect if a equals the sum of its positive divisors less than a.

1. True or False (Just answer true or false, you don't need to explain your answer):
   (a) [2 points] -23 is prime.

   (b) [2 points] $7|1001$.

   (c) [2 points] The sum of two odd numbers is odd.

   (d) [2 points] $T \subseteq A$ if and only if $T \in 2^A$.

   (e) [2 points] $\emptyset \subseteq \{\emptyset\}$.

   (f) [2 points] Let $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. n is a perfect number.

   (g) [2 points] $2 \in \{1, 2, \{1, 2\}\}$.

   (h) [2 points] If $x^2 < 0$, then x is a perfect number.

   (i) [2 points] Two right triangles that have hypotenuses of the same length have the same area.

   (j) [2 points] $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 1$.

2. For the following pairs of statements A, B, write a if the statement "If A, then B" is true, write b if the statement "If B, then A" is true and write c if the statement "A if and only if B is true". You should write all that apply.

   (a) [5 points] A: $x > 0$. B: $x^2 > 0$.

   (b) [5 points] A: Ellen is a grandmother. B: Ellen is female.

   (c) [5 points] A: x is odd. B: $x + 1$ is even.

   (d) [5 points] A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a square.

3. Proofs:

   (a) [5 points] Using the definition of *odd* integer provided in the "cheat sheet", prove that if n is an odd integer, then $-n$ is also an odd integer.

   (b) [5 points] Let a, b and d be integers. Suppose $b = aq + r$ where q and r are integers. Prove that if $d|a$ and $d|b$, then $d|r$.

4. Find counterexamples to disprove the following statements:

   (a) [5 points] An integer x is positive if and only if $x + 1$ is positive.

   (b) [5 points] An integer is a *palindrome* if it reads the same forwards and backwards when expressed in base 10. For example, 1331 is a *palindrome*. All *palindromes* are divisible by 11.

   (c) [5 points] If a, b and c are positive integers then $a^{(b^c)} = (a^b)^c$.

   (d) [5 points] Let A, B and C be sets. Then $A - (B - C) = (A - B) - C$.

5. Boolean Algebra

   (a) [5 points] Prove or disprove that the Boolean expressions $x \to \neg y$ and $\neg(x \to y)$ are logically equivalent.

   (b) [5 points] The expression $x \to y$ can be rewritten in terms of only the basic operations $\wedge, \vee$ and $\neg$; that is,$x \to y = (\neg x) \vee y$. Find an expression that is logically equivalent to $x \leftrightarrow y$ that uses only the operations $\wedge, \vee, \neg$ and prove that your expression is correct.

6. Consider the following proposition. Let N be a two-digit number and let M be the number formed from N by reversing the digits of N. Now compare N^2 and M^2. The digits of M^2 are precisely those of N^2, but reversed. For example:

$$10^2 = 100 \qquad 01^2 = 001$$

$$11^2 = 121 \qquad 11^2 = 121$$

$$12^2 = 144 \qquad 21^2 = 441$$

$$13^2 = 169 \qquad 31^2 = 961$$

and so on. Here is a proof of the proposition:
Proof. Since N is a two-digit number, we can write $N = 10a + b$ where a and b are the digits of N. Since M is formed from N by reversing digits, $M = 10b + a$.

Note that $N^2 = (10a + b)^2 = 100a^2 + 20ab + b^2 = (a^2) \times 100 + (2ab) \times 10 + (b^2) \times 1$, so the digits of N^2 are, in order, $a^2, 2ab, b^2$.

Likewise, $M^2 = (10b + a)^2 = (b^2) \times 100 + (2ab) \times 10 + (a^2) \times 1$, so the digits of M^2 are, in order, $b^2, 2ab, a^2$, exactly the reverse of N^2, which completes the proof.

(a) [5 points] Prove that the proposition is false.

(b) [5 points] Explain why the proof is invalid.

7. Counting

   (a) [5 points] In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ such that $a + b + c$ is even?

   (b) [5 points] Evaluate $\displaystyle\prod_{k=0}^{100} \frac{k^2}{k+1}$.

8. Let $A \times B = \{(1,2),(1,3),(1,7),(2,2),(2,3),(2,7)\}$.

(a) [5 points] What is $A \cup B$?

(b) [5 points] What is $A \cap B$?

(c) [5 points] What is $A - B$?

(d) [5 points] What is $A \Delta B$?

9. [10 points] Let A, B and C be sets. Prove that

$$(A \cup B) - C = (A - C) \cup (B - C).$$