# Homework 1
# Math 329: Number Theory

January 19, 2018

## 1 Easy

**Problem 1.** For each of the following pairs of integers, compute $\gcd(a, b)$ using the Euclidean algorithm.

(a) $a = 195, b = 90$.

(b) $a = 144, b = 89$.

**Problem 2.** For each of the following pairs of integers $a, b$, find integers $x$ and $y$ satisfying $ax+by = \gcd(a, b)$.

(a) $a = 195, b = 90$.

(b) $a = 144, b = 89$.

**Problem 3.** Let $a, b$ be two integers with the following properties:

- $a - b = 8$
- $(a, b)[a, b] = 384$.

Find the values of $a$ and $b$. *Hint : Use Problem 8.*

**Problem 4.** Let $n$ be a positive integer. Prove that $\gcd(n, n + 1) = 1$.

**Problem 5.** Prove that if $d|a$ and $d|b$, then $d|(ax + by)$ for any integers $x, y$.

## 2 Medium

**Problem 6.** Let $\{a_1, a_2, \ldots, a_{99}\}$ be a permutation of $\{1, 2, \ldots, 99\}$. Show that the product

$$(a_1 - 1)(a_2 - 2)\cdots(a_{99} - 99)$$

is even.

**Problem 7.** Suppose we perform the Euclidean algorithm on positive integers $a > b$. In the Euclidean algorithm on $a, b$, label the remainders $r_1 > r_2 > \ldots > r_k > 0$, where $r_k$ is the last nonzero remainder.

(a) Show that each nonzero remainder $r_m$ is less than $\frac{r_{m-2}}{2}$.

(b) Deduce that the number of divisions is at most $2n + 1$ where $n$ is the integer such that $2^n \leq b < 2^{n+1}$.

For the next two problems, we'll use the following notation:
Let $(a_1, a_2, \ldots, a_n)$ be the greatest common divisor of $a_1, a_2, \ldots, a_n$, and $[a_1, a_2, \ldots, a_n]$ be the least common multiple of $a_1, a_2, \ldots, a_n$.

**Problem 8.** Let $a, b$ be positive integers. Prove

$$(a, b)[a, b] = ab.$$

**Problem 9.** Let $a, b, c$ be positive integers. Prove

$$\frac{(a, b, c)^2}{(a, b)(b, c)(c, a)} = \frac{[a, b, c]^2}{[a, b][b, c][c, a]}.$$

## 3 Hard

**Problem 10.** Let $y > x > 1$ be integers such that $x + y \leq 100$. Alice is given $x + y$ (the sum of the two numbers) and Bob is given $xy$ (the product of the two numbers). They both know this set-up and that $y > x > 1, x + y \leq 100$. The following conversation occurs:

- Alice says "Bob does not know the values of $x$ and $y$."
- Bob says "Now I know the values of $x$ and $y$."
- Alice says "Now I also know the values of $x$ and $y$."

Assuming Alice and Bob are perfect logicians that don't lie. What are the values of $x$ and $y$?