# Homework 10
# Math 329: Number Theory

April 24, 2018

## 1 Easy

**Problem 1.** Let $\mu$ be the Möbius function. The definition of $\mu$ is in problem 6. Find the following values of $\mu$

(a) $\mu(105)$.

(b) $\mu(50)$.

(c) $\mu(2018)$.

**Problem 2.** Let $f(n)$ be the number of $j \leq n$ satisfy $\mu(j) = 0$. For example $f(10) = 3$ since $\mu(4) = \mu(9) = \mu(8) = 0$. What is $f(100)$?

**Problem 3.** Let $\sigma(n)$ be the sum of the divisors of $n$, for example $\sigma(6) = 1 + 2 + 3 + 6 = 12$. Find $\sigma(5^{100})$.

**Problem 4.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Find a formula for $\sigma(n)$. (You may assume $\sigma$ is multiplicative.)

**Problem 5.** We say that a number $n$ is perfect if it is the sum of its proper divisors.[1] For example, the proper divisors of 6 are 1,2,3 and $6 = 1 + 2 + 3$, so 6 is perfect. In other words, $n$ is perfect if $\sigma(n) = 2n$. Let $p$ be such that $2^p - 1$ is a prime number.[2] Prove that $n = 2^{p-1}(2^p - 1)$ is a perfect number.

## 2 Medium

**Problem 6.** Let $\mu$ be the Möbius function, i.e.,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ for distinct primes } p_1, p_2 \ldots p_k \\ 0 & \text{if there exists a positive integer } m \text{ such that } m^2 | n \end{cases}$$

Prove that $\mu$ is multiplicative.

**Problem 7.** Prove

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 8.** Let $\omega(n)$ be the number of distinct prime factors of $n$ and $\tau(n)$ be the number of positive divisors of $n$. Prove

$$\sum_{d|n} 2^{\omega(d)} = \tau(n^2).$$

[1] A proper divisor $d$ of $n$, is a divisor of $n$ smaller than $n$.

[2] primes of the form $2^p - 1$ are called Mersenne primes.

**Problem 9.** Let $\lambda(1) = 1$ and $\lambda(n) = (-1)^{\alpha_1+\alpha_2+\ldots+\alpha_r}$ if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$.

(a) Prove that $\lambda$ is totally multiplicative.

(b) Prove that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

# 3 Hard

**Problem 10.** For each positive integer $n$, let $\tau(n)$ be the number of divisors of $n$. Find all positive integers such that

$$n + \tau(n) = (\tau(n))^2.$$

**Problem 11.** Let $q$ be an odd positive integer, and let $N_q$ denote the number of integers $a$ such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that $N_q$ is odd if and only if $q$ is of the form $p^k$ with $k$ a positive integer and $p$ a prime congruent to 5 or 7 modulo 8.

**Problem 12.** Let $f(n)$ be the number of remainders $a$ modulo $10^n$ for which there exists an integer $x$ such that $x^2 \equiv a \bmod 10^n$. For example, when $n = 1$, we have that $\{0, 1, 4, 5, 6, 9\}$ are the remainders modulo 10 for which there is an integer $x$ satisfying $x^2 \equiv a \bmod 10$. Therefore $f(1) = 6$.

Find

$$\lim_{n\to\infty} \frac{f(n)}{10^n}.$$