Homework 2
Math 329: Number Theory

January 26, 2018

1 Easy

Problem 1. Let $q_1 = 2$. Then for $n \geq 2$. define q_n to be the **smallest** prime factor of $q_1 q_2 \cdots q_{n-1} + 1$. Find the first ten elements of the sequence, i.e., find $q_1, q_2, \ldots, q_{10}$.

Problem 2. Let $q_1 = 2$. Then for $n \geq 2$. define q_n to be the **largest** prime factor of $q_1 q_2 \cdots q_{n-1} + 1$. Find the first ten elements of the sequence, i.e., find $q_1, q_2, \ldots, q_{10}$.

Problem 3. Suppose that n is the product of three consecutive integers and that n is divisible by 7. Which of the following is not necessarily a divisor of n? Why?
(a) 6 (b) 14 (c) 21 (d) 28 (e) 42

Problem 4. Prove that if n is composite, it has a prime factor not exceeding $\sqrt{n}$.

Problem 5. Show that if $b|a$ and $c|a$ with $(b, c) = 1$, then $bc|a$.

2 Medium

Problem 6. Let r be a positive integer. Prove that

$$\underbrace{11 \cdots 1}_{2r} - \underbrace{22 \cdots 2}_{r}$$

is a square.

Problem 7. Prove that there are infinitely many primes of the form $6k - 1$.

Problem 8. Let $x \geq 7$ be a real number. Suppose there are only n primes, say $p_1, p_2, \ldots p_n$.

(a) Show that this would imply that there are at most $(2 \log x)^n$ numbers between 1 and x. Note: We use log to refer to the natural logarithm, i.e., $\log(e) = 1$.

(b) Show that (a) implies that there are infinitely many primes.

Problem 9. Show that the number

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not an integer for any integer $n > 1$.

3 Hard

Problem 10. For a pair of positive integers a and b that are not multiples of 5, the following list is constructed: The first number is 5 and, starting with the second number, each number is obtained by multiplying the previous number on the list by a and then adding b. What is the maximum number of primes that the list can have before obtaining the first composite number?