Homework 3
Math 329: Number Theory

February 1, 2018

1 Easy

Problem 1. Find an integer $0 \leq r < 997$ satisfying $r \equiv 2^{1000} \bmod 997$.

Problem 2. Prove that $2011^{1001} + 2012^{1001} + 2013^{1001} + 2014^{1001} + 2015^{1001} + 2016^{1001} + 2017^{1001}$ is a multiple of 7.

Problem 3. Show that if n is odd, then $n^2 \equiv 1 \bmod 8$.

Problem 4. Find an integer $0 \leq r < 977$ such that $r \equiv 2^{972} \bmod 977$.

Problem 5. Prove that every palindrome with an even number of digits is a multiple of 11. Note: A number is a palindrome if it can be read from left-to-right or right-to-left and yield the same number. For example, 121, 5445 and 12521 are palindromes.

2 Medium

Problem 6. How many 40-digit palindromes are there that are multiples of 99?

Problem 7. Find all integers n with 7 digits where every digit is 3 or 7, and that satisfy that $3|n$ and $7|n$.

Problem 8. Let $S = \{9, 99, 999, \ldots\}$. Prove that for any prime $p > 5$, there are infinitely many members of S that are divisible by p.

Problem 9. The sum of the squares of two consecutive numbers can be equal a perfect square, namely $3^2 + 4^2 = 5^2$.

(a) Show that the sum of the squares of m consecutive integers cannot be a square if $m = 3$ or $m = 6$.

(b) Find an example of 11 consecutive integers where the sum of its squares is a square.

3 Hard

Problem 10. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \qquad n \geq 1.$$

Note: We say that a and b are relatively prime if $(a, b) = 1$.