Homework 4
Math 329: Number Theory

February 14, 2018

1 Easy

Problem 1. List the quadratic residues modulo 23.

Problem 2. For the following numbers n, calculate $\phi(n) = |\{k \leq n \mid \gcd(k, n) = 1\}|$:

(a) $n = 40$.

(b) $n = 210$.

Problem 3. For the following numbers n, calculate $\sigma(n)$, the sum of the divisors of n:

(a) $n = 40$.

(b) $n = 210$.

Problem 4. Compute the last three digits of 57^{403}.

Problem 5. Compute $3^{300} \bmod 343$.

2 Medium

Problem 6. Show that for an integer $n > 1$, the sum of the positive integers less than n that are relatively prime to n is $\frac{n\phi(n)}{2}$. In other words, prove

$$\sum_{\substack{k \leq n \\ (k,n)=1}} k = \frac{n\phi(n)}{2}.$$

Problem 7. Let n be positive.

(a) Show that any solution of the equation $\phi(x) = 4n + 2$ is of the form $x = p^\alpha$ or $x = 2p^\alpha$ for some prime p of the form $4s - 1$.

(b) Deduce that there are no solutions to the equation $\phi(x) = 14$.

Problem 8. Find all primes p such that $13^{2p-1} + 17$ is divisible by p.

Problem 9. Let n be a positive integer. Show that there is a power of 3 greater than 1 whose final n digits are $\underbrace{00\cdots0}_{n-1}1$.

3 Hard

Problem 10. Prove that for each positive integer n, the number

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$

is not prime.