Homework 5
Math 329: Number Theory

February 19, 2018

1 Easy

Problem 1. Solve the system of congruences:

$$x \equiv 4 \bmod 10$$
$$x \equiv 8 \bmod 13.$$

Problem 2. Solve the system of congruences:

$$3x \equiv 5 \bmod 7$$
$$7x \equiv 5 \bmod 15.$$

Problem 3. Solve the system of congruences:

$$x \equiv 1 \bmod 2$$
$$x \equiv 2 \bmod 3$$
$$x \equiv 3 \bmod 5$$
$$x \equiv 5 \bmod 7.$$

Problem 4. Solve the system of congruences:

$$7x \equiv 7 \bmod 21$$
$$3x \equiv 6 \bmod 12$$
$$6x \equiv 8 \bmod 14.$$

Problem 5. A fruit vendor wants to know how many oranges he had yesterday. He remembers the count was between 100 and 150 and he remembers that if it was grouped in pairs, triples, quadruples, quintuples or sextuples, there was always an extra orange. How many oranges did the vendor have?

2 Medium

Problem 6. What are the last three digits of 35^{652}?

Problem 7. A famous theorem by Dirichlet asserts that if $(k, \ell) = 1$, there are infinitely many primes of the form $kx + \ell$. Prove the weaker statement that if $(k, \ell) = 1$ and $n \neq 0$, there are infinitely many integers x such that $(kx + \ell, n) = 1$.
Hint: Consider a prime factor p of n. Figure out what congruence of x mod p forces $kx+\ell$ to not be divisible by p.

Problem 8. We say that a number n is *squarefree* if it is not divisible by any square greater than 1. For example, 6 is squarefree because 6 is not divisible by any square other than 1, while 8 is not squarefree because it is divisible by 4.

Prove that for any positive integer n, there exist n consecutive numbers that are **not** squarefree.
Hint: Build a system of n congruence equations that force n consecutive integers to have a square divisor.

Problem 9. How many solutions modulo 2002 are there to the equation $a^{2002} - a \equiv 0 \bmod 2002$?

3 Hard

Problem 10. Five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night.

But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had one coconut left over, and gave it to the monkey, and he hid his pile and put the rest back together.

By and by, the next man woke up and did the same thing. And he had one left over and he gave it to the monkey. And all five of the men did the same thing, one after the other; each one taking the fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in the morning they divided what coconuts were left, and they came out in five equal shares. Of course each one must have known that there were coconuts missing; but each one was guilty as the others, so they didn't say anything. How many coconuts where there in the beginning?