Homework 6
Math 329: Number Theory

March 2, 2018

1 Easy

Problem 1. Find all solutions in **nonnegative integers** to each equation here:

(a) $3x + 4y = 12$.

(b) $3x + 4y = 23$.

Problem 2. Find all solutions in **nonnegative integers** to each equation here:

(a) $16x + 28y = 356$.

(b) $18x + 31y = 491$.

Problem 3. Find all integer solutions to the following linear Diophantine equations.

(a) $6x + 11y = 2$.

(b) $15x - 64y = 103$.

Problem 4. Can 1000 be expressed as the sum of two integers, one of which is divisible by 11 and the other by 17? If so, determine one such way.

Problem 5. In each case below, determine how many positive integers k will permit no solutions in nonnegative integers (x, y) at all:

(a) $5x + 3y = k$.

(b) $15x + 7y = k$.

2 Medium

Problem 6. A person cashes a cheque at the bank. By mistake the teller pays the person the number of cents as dollars and the number of dollars as cents. The person spends \$3.50 before noticing the mistake, then after counting the money finds that there is exactly double the amount of the cheque. For what amount was the cheque drawn?

Problem 7. A player scores either A or B at each turn, where A and B are unequal positive integers. He notices that his cumulative score can take any positive integer value except for those in a finite set S, where $|S| = 35$ and $58 \in S$. Find A and B.

Problem 8. Prove that if a, b, c are integers and are the side lengths of a right triangle, then $60|abc$.

Problem 9. Prove that there are no positive integers a, b such that $a^2 + b^2$ and $a^2 - b^2$ are both squares.

3 Hard

Problem 10. Suppose that $f : \{1, 2, \ldots, 1600\} \to \{1, 2, \ldots, 1600\}$ satisfies $f(1) = 1$ and

$$f^{2005}(x) = x \qquad \text{for } \; x = 1, 2, \ldots, 1600.$$

(a) Prove that f has a fixed point different from 1.

(b) Find all $n \geq 1600$ such that any $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying the above condition has at least two fixed points.

Notes:

- In the problem we use the convention that $f^i(x)$ means composing f with itself i times. For example $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, and so on.
- We say that y is a fixed point of f if $f(y) = y$.