Homework 9
Math 329: Number Theory

April 18, 2018

1 Easy

Problem 1. Compute $\left(\frac{10}{19}\right)$ using Gauss' lemma.

Problem 2. Compute $\left(\frac{7}{11}\right)$ using Eisenstein's lemma.

Problem 3. Evaluate $\left(\frac{2008}{257}\right)$.

Problem 4. Evaluate $\left(\frac{503}{773}\right)$ and $\left(\frac{501}{773}\right)$.

Problem 5. Using formulas for $\left(\frac{2}{p}\right)$ and $\left(\frac{-1}{p}\right)$, find a general formula for $\left(\frac{-2}{p}\right)$.

2 Medium

Problem 6. Let $g(p)$ be the least quadratic non-residue modulo p (where p is an odd prime). Prove that $g(p)$ is prime.

Problem 7. Prove that if p is an odd prime, then

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \bmod 12 \\ -1 & \text{if } p \equiv \pm 5 \bmod 12. \end{cases}$$

Problem 8. Let $g(p)$ be the least quadratic non-residue modulo p (where p is an odd prime). For example, for $p = 7$, we have $g(7) = 3$ because 1 and 2 are quadratic residues, while 3 is not a quadratic residue, so 3 is the least non-residue. Prove that

$$g(p) = 5 \Leftrightarrow p \equiv \pm 23, \pm 47 \bmod 120.$$

Problem 9. Let $q_1 = 2$. Then for $n \geq 2$. define q_n to be the **largest** prime factor of $q_1 q_2 \cdots q_{n-1} + 1$. For example, the first seven elements of the sequence are $2, 3, 7, 43, 139, 50207, 340999$. Prove that 5 is not on the sequence.

3 Hard

Problem 10. Let $g(p)$ be the least quadratic non-residue modulo p (where p is an odd prime). Prove that $g(p) < \sqrt{p} + 1$.