Homework 1

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Most problems below are from Judson.

1. Find all of the ideals in each of the following rings. Which of these ideals are maximal and which are prime?
   (a) $\mathbb{Z}_{18}$
   (b) $\mathbb{Z}_{25}$
   (c) $\mathbb{Q}$
2. Find all ring homomorphisms $\phi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$.
3. Let m, n be positive integers. How many ring homomorphisms are there from $\mathbb{Z}_m$ to $\mathbb{Z}_n$? Hint: Consider $d = \gcd(m, n)$.
4. Prove or disprove: The ring $\mathbb{Q}(\sqrt{2}\,) = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$ is isomorphic to the ring $\mathbb{Q}(\sqrt{3}\,) = \{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$.
5. Prove that the Gaussian integers, $\mathbb{Z}[i]$, are an integral domain.
6. Let $\phi : R \to S$ be a ring homomorphism. Prove each of the following statements.
   (a) If R is a commutative ring, then $\phi(R)$ is a commutative ring.
   (b) $\phi(0) = 0$.
   (c) Let 1_R and 1_S be the identities for R and S, respectively. If ϕ is onto, then $\phi(1_R) = 1_S$.
   (d) If R is a field and $\phi(R) \neq 0$, then $\phi(R)$ is a field.
7. Prove the Third Isomorphism Theorem for rings: Let R be a ring and I and J be ideals of R, where $J \subset I$. Then
$$R/I \cong \frac{R/J}{I/J}.$$
8. Let R be an integral domain. Show that if the only ideals in R are $\{0\}$ and R itself, R must be a field.
9. Let R be a commutative ring. An element a in R is **nilpotent** if $a^n = 0$ for some positive integer n. Show that the set of all nilpotent elements forms an ideal in R.
10. Let p be prime. Prove that
$$\mathbb{Z}_{(p)} = \{a/b : \ a, b \in \mathbb{Z} \text{ and } \gcd(b, p) = 1\}$$
is a ring. The ring $\mathbb{Z}_{(p)}$ is called the **ring of integers localized at** p.