Homework 2

Most problems below are from Judson.

1. Compute each of the following.
   (a) $(3x^2 + 2x - 4) + (4x^2 + 2)$ in $\mathbb{Z}_5$
   (b) $(3x^2 + 2x - 4)(4x^2 + 2)$ in $\mathbb{Z}_5$
   (c) $(5x^2 + 3x - 2)^2$ in $\mathbb{Z}_{12}$
2. Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$ for each of the following pairs of polynomials.
   (a) $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1$ and $b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$
   (b) $a(x) = 4x^5 - x^3 + x^2 + 4$ and $b(x) = x^3 - 2$ in $\mathbb{Z}_5[x]$
   (c) $a(x) = x^5 + x^3 - x^2 - x$ and $b(x) = x^3 + x$ in $\mathbb{Z}_2[x]$
3. Find all of the zeros for each of the following polynomials.
   (a) $5x^3 + 4x^2 - x + 9$ in $\mathbb{Z}_{12}$
   (b) $3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5$
   (c) $5x^4 + 2x^2 - 3$ in $\mathbb{Z}_7$
   (d) $x^3 + x + 1$ in $\mathbb{Z}_2$
4. Find a unit $p(x)$ in $\mathbb{Z}_4[x]$ such that $\deg p(x) > 1$.
5. Which of the following polynomials are irreducible over $\mathbb{Q}[x]$?
   (a) $x^4 - 2x^3 + 2x^2 + x + 4$
   (b) $x^4 - 5x^3 + 3x - 2$
   (c) $3x^5 - 4x^3 - 6x^2 + 6$
   (d) $5x^5 - 6x^4 - 3x^2 + 9x - 15$
6. Let $f(x)$ be irreducible. If $f(x) \mid p(x)q(x)$, prove that either $f(x) \mid p(x)$ or $f(x) \mid q(x)$.
7. **The Rational Root Theorem.** Let
   $$p(x) = a_n x^n a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x],$$
   where $a_n \neq 0$. Prove that if $p(r/s) = 0$, where $\gcd(r, s) = 1$, then $r \mid a_0$ and $s \mid a_n$.
8. **Cyclotomic Polynomials.** The polynomial
   $$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$
   is called the *cyclotomic polynomial*. Show that $\Phi_p(x)$ is irreducible over $\mathbb{Q}$ for any prime p.
9. Let $p(x)$ and $q(x)$ be polynomials in $R[x]$, where R is a commutative ring with identity. Prove that $\deg(p(x) + q(x)) \leq \max(\deg p(x), \deg q(x))$.
10. We call a polynomial $p(x) \in \mathbb{Z}_2[x]$ perfect if the sum of its divisors $\sigma(p(x))$ equals $p(x)$. For example $x^2 + x$ is perfect because $\sigma(x^2 + x) = 1 + x + (x + 1) + (x^2 + x) = x^2 + x \bmod 2$. Suppose $p(x)$ is perfect. Prove that $x|p(x)$ if and only if $(x + 1)|p(x)$.