Homework 3

Most problems below are from Judson.

1. The Gaussian integers, $\mathbb{Z}[i]$, are a UFD. Factor each of the following elements in $\mathbb{Z}[i]$ into a product of irreducibles.
   (a) 5
   (b) $1 + 3i$
   (c) $6 + 8i$
   (d) 2
2. Let D be an integral domain.
   (a) Prove that F_D is an abelian group under the operation of addition.
   (b) Show that the operation of multiplication is well-defined in the field of fractions, F_D.
   (c) Verify the associative and commutative properties for multiplication in F_D.
3. Prove or disprove: Any subring of a field F containing 1 is an integral domain.
4. Prove or disprove: If D is an integral domain, then every prime element in D is also irreducible in D.
5. Let p be prime and denote the field of fractions of $\mathbb{Z}_p[x]$ by $\mathbb{Z}_p(x)$. Prove that $\mathbb{Z}_p(x)$ is an infinite field of characteristic p.
6. Let $\mathbb{Z}[\sqrt{2}\,] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.
   (a) Prove that $\mathbb{Z}[\sqrt{2}\,]$ is an integral domain.
   (b) Find all of the units in $\mathbb{Z}[\sqrt{2}\,]$.
   (c) Determine the field of fractions of $\mathbb{Z}[\sqrt{2}\,]$.
   (d) Prove that $\mathbb{Z}[\sqrt{2}i]$ is a Euclidean domain under the Euclidean valuation $\nu(a + b\sqrt{2}\,i) = a^2 + 2b^2$.
7. Let D be a Euclidean domain with Euclidean valuation ν. If u is a unit in D, show that $\nu(u) = \nu(1)$.
8. An ideal of a commutative ring R is said to be **finitely generated** if there exist elements $a_1, \ldots, a_n$ in R such that every element $r \in R$ can be written as $a_1 r_1 + \cdots + a_n r_n$ for some $r_1, \ldots, r_n$ in R. Prove that R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.
9. Let R be a PID. Let P be a prime ideal of R. Prove that R/P is a PID.
10. (a) Prove that $\mathbb{Z}[i]/\langle 1 + i\rangle$ is a field of order 2.
    (b) Let $q \in \mathbb{Z}$ be a prime with $q \equiv 3 \bmod 4$. Prove that $\mathbb{Z}[i]/\langle q\rangle$ is a field with q^2 elements.