Homework 4

Most problems below are from Judson.

1. If F is a field, show that $F[x]$ is a vector space over F, where the vectors in $F[x]$ are polynomials. Vector addition is polynomial addition, and scalar multiplication is defined by $\alpha p(x)$ for $\alpha \in F$.

2. Let $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$ be the field generated by elements of the form $a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are in $\mathbb{Q}$. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$ is a vector space of dimension 4 over $\mathbb{Q}$. Find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$.

3. Let F be a field and denote the set of n-tuples of F by F^n. Given vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in F^n and α in F, define vector addition by

$$u + v = (u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$$

   and scalar multiplication by

$$\alpha u = \alpha(u_1, \ldots, u_n) = (\alpha u_1, \ldots, \alpha u_n).$$

   Prove that F^n is a vector space of dimension n under these operations.

4. Which of the following sets are subspaces of $\mathbb{R}^3$? If the set is indeed a subspace, find a basis for the subspace and compute its dimension.

   (a) $\{(x_1, x_2, x_3) : 3x_1 - 2x_2 + x_3 = 0\}$

   (b) $\{(x_1, x_2, x_3) : 3x_1 + 4x_3 = 0, 2x_1 - x_2 + x_3 = 0\}$

   (c) $\{(x_1, x_2, x_3) : x_1 - 2x_2 + 2x_3 = 2\}$

   (d) $\{(x_1, x_2, x_3) : 3x_1 - 2x_2^2 = 0\}$

5. Let V be a vector space of dimension n. Prove each of the following statements.

   (a) If $S = \{v_1, \ldots, v_n\}$ is a set of linearly independent vectors for V, then S is a basis for V.

   (b) If $S = \{v_1, \ldots, v_n\}$ spans V, then S is a basis for V.

   (c) If $S = \{v_1, \ldots, v_k\}$ is a set of linearly independent vectors for V with $k < n$, then there exist vectors $v_{k+1}, \ldots, v_n$ such that

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

   is a basis for V.

6. Prove that any set of vectors containing $\mathbf{0}$ is linearly dependent.

7. If a vector space V is spanned by n vectors, show that any set of m vectors in V must be linearly dependent for $m > n$.

8. **Linear Transformations.** Let V and W be vector spaces over a field F, of dimensions m and n, respectively. If $T : V \to W$ is a map satisfying

$$T(u + v) = T(u) + T(v)$$
$$T(\alpha v) = \alpha T(v)$$

   for all $\alpha \in F$ and all $u, v \in V$, then T is called a **linear transformation** from V into W.

(a) Prove that the **kernel** of T, $\ker(T) = \{v \in V : T(v) = \mathbf{0}\}$, is a subspace of V. The kernel of T is sometimes called the **null space** of T.

(b) Prove that the **range** or **range space** of T, $R(V) = \{w \in W : T(v) = w \text{ for some } v \in V\}$, is a subspace of W.

(c) Show that $T : V \to W$ is injective if and only if $\ker(T) = \{\mathbf{0}\}$.

(d) Let $\{v_1, \ldots, v_k\}$ be a basis for the null space of T. We can extend this basis to be a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ of V. Why? Prove that $\{T(v_{k+1}), \ldots, T(v_m)\}$ is a basis for the range of T. Conclude that the range of T has dimension $m - k$.

(e) Let $\dim V = \dim W$. Show that a linear transformation $T : V \to W$ is injective if and only if it is surjective.

9. Let V and W be finite dimensional vector spaces of dimension n over a field F. Suppose that $T : V \to W$ is a vector space isomorphism. If $\{v_1, \ldots, v_n\}$ is a basis of V, show that $\{T(v_1), \ldots, T(v_n)\}$ is a basis of W. Conclude that any vector space over a field F of dimension n is isomorphic to F^n.