Homework 4 Solutions

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Most problems below are from Judson.

1. If F is a field, show that $F[x]$ is a vector space over F, where the vectors in $F[x]$ are polynomials. Vector addition is polynomial addition, and scalar multiplication is defined by $\alpha p(x)$ for $\alpha \in F$.

   Solution 1. Let $p(x), q(x) \in F[x]$. Then $p(x) + q(x)$ is also a polynomial, so it's also in $F[x]$. The scalar multiple of $\alpha p(x)$ is also a polynomial with coefficients in F. Therefore $F[x]$ is a vector space over F.

2. Let $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$ be the field generated by elements of the form $a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are in $\mathbb{Q}$. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$ is a vector space of dimension 4 over $\mathbb{Q}$. Find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3}\,)$.

   Solution 2. A basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $\mathfrak{B} = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

   We'll prove that the span of $\mathfrak{B}$ is indeed $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ to show that it is a vector space. We'll first prove that $\mathfrak{B}$ is linearly independent.

   Suppose there are $c_1, c_2, c_3, c_4 \in \mathbb{Q}$ not all zero such that

   $$c_1 + c_2\sqrt{2} + c_3\sqrt{3} + c_4\sqrt{6} = 0.$$

   Then (after rearranging and squaring)

   $$\begin{aligned} c_2\sqrt{2} + c_3\sqrt{3} &= -c_1 - c_4\sqrt{6} \\ 2c^2 + 3c_3^2 + 2c_2c_3\sqrt{6} &= c_1^2 + 6c_4^2 + 2c_1c_4\sqrt{6} \\ 2c_2^2 + 3c_3^2 - c_1^2 - 6c_4^2 &= (2c_1c_4 - 2c_2c_3)\sqrt{6}. \end{aligned}$$

   Note that the left side is rational and that $2c_1c_4 - 2c_2c_3$ is rational. If $2c_1c_4 - 2c_2c_3 \neq 0$, then $\sqrt{6} \in \mathbb{Q}$, which is a contradiction.

   Therefore $2c_1c_4 - 2c_2c_3 = 0$, so $2c_2^2 + 3c_3^2 - c_1^2 - 6c_4^2 = 0$ as well. Since we can scale by any positive integer, we may assume c_1, c_2, c_3, c_4 are integers with $\gcd(c_1, c_2, c_3, c_4) = 1$. If $c_4 = 0$, then $c_2c_3 = 0$ so $c_2 = 0$ or $c_3 = 0$. When $c_2 = 0$ we have $c_3\sqrt{3} = -c_1$ which implies that $c_1 = c_3 = 0$ or that $\sqrt{3} \in \mathbb{Q}$. Both are not possible. In the case $c_3 = 0$ we have $\sqrt{2} \in \mathbb{Q}$ or $c_1 = c_2 = c_3 = c_4 = 0$. Therefore we may assume $c_4 \neq 0$.

   We have $2c_2^2 + 3c_3^2 = c_1^2 + 6c_4^2$ with all integers and $\gcd(c_1, c_2, c_3, c_4) = 1$. Then modulo 3 we have $2c_2^2 \equiv c_1^2 \bmod 3$. But $x^2 \equiv 0, 1, \bmod 3$, therefore $2c_2^2 \equiv 0, 2 \bmod 3$ and $c_1^2 \equiv 0, 1 \bmod 3$, so $c_2 \equiv c_1 \equiv 0 \bmod 3$. Therefore $c_2 = 3c_2', c_1 = 3c_1'$. But then

   $$\begin{aligned} 18c_2'^2 + 3c_3^2 &= 9c_1'^2 + 6c_4^2 \\ 6c_2'^2 + c_3^2 &= 3c_1'^2 + 2c_4^2. \end{aligned}$$

   But then $c_3^2 \equiv 2c_4^2 \bmod 3$. Therefore, $c_3 \equiv c_4 \bmod 3$. But then c_1, c_2, c_3, c_4 are all multiples of 3, contradicting that $\gcd(c_1, c_2, c_3, c_4) = 1$. Therefore, $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is linearly independent.

   Therefore $\text{Span}\{1, \sqrt{2}, \sqrt{3}\}\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \text{Span}\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

3. Let F be a field and denote the set of n-tuples of F by F^n. Given vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in F^n and α in F, define vector addition by

$$u + v = (u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n)$$

and scalar multiplication by

$$\alpha u = \alpha(u_1, \dots, u_n) = (\alpha u_1, \dots, \alpha u_n).$$

Prove that F^n is a vector space of dimension n under these operations.

Solution 3. To check closed under addition: Since $u_i + v_i \in F$, for all $i = 1, 2, \dots, n$, then $u + v$ is in F^n.

Since $\alpha u_i \in F$ (because $\alpha \in F$), then $\alpha u \in F^n$. Therefore F^n is a vector space over F.

Now we need to verify the dimension. We need to show $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ with the 1 being in the i-th position form a basis for F^n. Suppose

$$\sum_{i=1}^{n} c_i e_i = 0.$$

Then the vector $(c_1, c_2, \dots, c_n)$ is zero. But that means $c_1, c_2, \dots, c_n$ are all zero. Therefore the e_i's are linearly independent. Now for any vector u we have

$$u = \sum_{i=1}^{n} u_i e_i.$$

Therefore u is in the span of the e_i's. Therefore we have a basis of n elements.

4. Which of the following sets are subspaces of $\mathbb{R}^3$? If the set is indeed a subspace, find a basis for the subspace and compute its dimension.

(a) $\{(x_1, x_2, x_3) : 3x_1 - 2x_2 + x_3 = 0\}$
(b) $\{(x_1, x_2, x_3) : 3x_1 + 4x_3 = 0, 2x_1 - x_2 + x_3 = 0\}$
(c) $\{(x_1, x_2, x_3) : x_1 - 2x_2 + 2x_3 = 2\}$
(d) $\{(x_1, x_2, x_3) : 3x_1 - 2x_2^2 = 0\}$

Solution 4.

(a) It is a subspace. Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3)$ are in the set. Then $3x_1 - 2x_2 + x_3 = 0$ and $3y_1 - 2y_2 + y_3 = 0$, so $(3(x_1 + y_1) - 2(x_2 + y_2) + (x_3 + y_3) = 0$, so $(x_1 + y_1, x_2 + y_2, x_3 + y_3$ is in the set. With respect to scalar multiple, $3(\alpha x_1) - 2(\alpha x_2) + \alpha x_3 = \alpha(3x_1 - 2x_2 + x_3) = 0$.

To find the dimension. Note that $x_3 = -3x_1 + 2x_2$. Once x_1, x_2 are selected, x_3 is fixed. The dimension is ≤ 2. It cannot be one, since not all terms are multiples of one of them. Therefore, the dimension is 2.

Another way to solve this is to see that $T((x_1, x_2, x_3)) = 3x_1 - 2x_2 + x_3$ is a linear transformation. This set is the kernel, so it is a subspace. The dimension of the kernel is 3 minus the dimension of the range. But the range is $\mathbb{R}$, so it has dimension 1. Therefore, the kernel has dimension 2.

(b) It is a subspace. Indeed, suppose $(x_1, x_2, x_3), (y_1, y_2, y_3)$ are in the set and α is a scalar. We know $3x_1 + 4x_3 = 0$ and $2x_1 - x_2 + x_3 = 0$, so $3(\alpha x_1) + 4(\alpha x_3) = 0$ and $2(\alpha x_1) - (\alpha x_2) + (\alpha x_3) = 0$. Also

$$3(x_1 + y_1) + 4(x_3 + y_3) = (3x_1 + 4x_3) + (3y_1 + 4y_3) = 0,$$

and

$$2(x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3) = (2x_1 - x_2 + x_3) + (2y_1 - y_2 + y_3) = 0.$$

For the dimension, note that $x_1 = -\frac{4}{3}x_3$ and that

$$x_2 = 2x_1 + x_3 = -\frac{8}{3}x_3 + x_3 = -\frac{5}{3}x_3.$$

Therefore, once x_3 is chosen, then x_1, x_2 are fixed. The dimension is therefore 1.

(c) It is not a subspace. Note $(0,0,2)$ and $(0,-1,0)$ are in the set but $(0,-1,2)$ is not.

(d) Note $(2/3,1,0)$ and $(2/3,-1,0)$ are in the set. Yet their sum is $(4/3,0,0)$ is not in the set since $3(4/3)-2(0)^2=4\neq 0$.

5. Let V be a vector space of dimension n. Prove each of the following statements.

   (a) If $S=\{v_1,\ldots,v_n\}$ is a set of linearly independent vectors for V, then S is a basis for V.

   (b) If $S=\{v_1,\ldots,v_n\}$ spans V, then S is a basis for V.

   (c) If $S=\{v_1,\ldots,v_k\}$ is a set of linearly independent vectors for V with $k<n$, then there exist vectors $v_{k+1},\ldots,v_n$ such that
   $$\{v_1,\ldots,v_k,v_{k+1},\ldots,v_n\}$$
   is a basis for V.

   Solution 5.

   (a) Suppose it was not a basis. Then there exists $v\in V$ such that v is not in the span of $\{v_1,v_2,\cdots,v_n\}$. Therefore $\{v_1,\cdots,v_n,v\}$ is a linearly independent set. But this is impossible (as proved in problem 7).

   (b) Suppose it's not linearly independent. Then
   $$v_n=\sum_{i=1}^{n-1}c_iv_i,$$
   for some $c_i\in\mathbb{F}$. But then $\{v_1,v_2,\ldots,v_{n-1}\}$ is a span of V. Now consider the basis of V with n vectors. As proved in problem 7, this basis would be linearly dependent. That's a contradiction. Therefore $\{v_1,\cdots,v_n\}$ is linearly independent.

   (c) The proof of this is the construction described in the solution of problem 7.

6. Prove that any set of vectors containing $\mathbf{0}$ is linearly dependent.

   Solution 6. Let $\{v_1,v_2,\ldots,v_r\}$ be a set of vector containing 0. Without loss of generality, let $v_1=0$. Then
   $$v_1+0v_2+\cdots+0v_r=0.$$
   But the coefficient of $v_1\neq 0$, therefore the set is linearly dependent.

7. If a vector space V is spanned by n vectors, show that any set of m vectors in V must be linearly dependent for $m>n$.

   Solution 7. Let $\{v_1,\ldots,v_n\}$ span V and let $\{u_1,\ldots,u_m\}$ be m vectors in V. Suppose for the sake of contradiction that $\{u_1,u_2,\ldots,u_m\}$ is linearly independent. Since $\{v_1,\ldots,v_n\}$ spans V, then
   $$u_1=\sum_{i=1}^{n}c_iv_i.$$
   If $u_1=0$, then $\{u_1,\ldots,u_m\}$ is not linearly independent, so at least one of the c_i's is nonzero. Reorder the vectors $\{v_1,\ldots,v_n\}$ to force $c_1\neq 0$. Therefore
   $$v_1=\frac{1}{c_1}\left(u_1-c_2v_2-c_3v_3-\cdots-c_nv_n\right).$$
   Therefore v_1 is in the span of $\{u_1,v_2,v_3,\ldots,v_n\}$. But since $v_2,v_3,\ldots,v_n$ are also in that span, then $\{u_1,v_2,\ldots,v_n\}$ spans all of V. Therefore, there exist integers $c_1,c_2,\ldots,c_n$ such that
   $$u_2=c_1u_1+c_2v_2+c_3v_3+\cdots+c_nv_n.$$

Note that since $\{u_1, u_2\}$ are independent, then $c_2v_2 + \cdots + c_nv_n \neq 0$. Therefore, there's at least one nonzero coefficient. Reordering, we can force that coefficient to be $c_2 \neq 0$. But then

$$v_2 = \frac{1}{c_2}\left(u_2 - c_1u_1 - c_3v_3 - \cdots - c_nv_n\right).$$

Therefore v_2 is in the span of $\{u_1, u_2, v_3, \cdots, v_n\}$. As before, this implies $\{u_1, u_2, v_3, v_4, \cdots, v_n\}$ spans V. Therefore we can write u_3 in terms of these vectors and by similar reasoning conclude that $\{u_1, u_2, u_3, v_4, v_5, \cdots, v_n\}$ spans V. We can continue inductively to eventually show $\{u_1, u_2, \ldots, u_n\}$ spans V. We have $m > n$, so $m \geq n + 1$. Consider u_{n+1}. Since $\{u_1, \ldots, u_n\}$ spans V, then

$$u_{n+1} = c_1u_1 + c_2u_2 + \cdots + c_nu_n.$$

Therefore $\{u_1, u_2, \ldots, u_n, u_{n+1}\}$ is linearly dependent. Contradiction!

Therefore $\{u_1, u_2, \ldots, u_m\}$ is linearly dependent.

8. **Linear Transformations.** Let V and W be vector spaces over a field F, of dimensions m and n, respectively. If $T : V \to W$ is a map satisfying

$$T(u + v) = T(u) + T(v)$$
$$T(\alpha v) = \alpha T(v)$$

for all $\alpha \in F$ and all $u, v \in V$, then T is called a **linear transformation** from V into W.

(a) Prove that the **kernel** of T, $\ker(T) = \{v \in V : T(v) = \mathbf{0}\}$, is a subspace of V. The kernel of T is sometimes called the **null space** of T.

(b) Prove that the **range** or **range space** of T, $R(V) = \{w \in W : T(v) = w \text{ for some } v \in V\}$, is a subspace of W.

(c) Show that $T : V \to W$ is injective if and only if $\ker(T) = \{\mathbf{0}\}$.

(d) Let $\{v_1, \ldots, v_k\}$ be a basis for the null space of T. We can extend this basis to be a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ of V. Why? Prove that $\{T(v_{k+1}), \ldots, T(v_m)\}$ is a basis for the range of T. Conclude that the range of T has dimension $m - k$.

(e) Let $\dim V = \dim W$. Show that a linear transformation $T : V \to W$ is injective if and only if it is surjective.

Solution 8.

(a) Let $u, w \in \ker(T)$ and α is a scalar. Since $u, w \in T$, then $T(u) = T(w) = 0$. Since T is a linear transformation, then

$$T(u + w) = T(u) + T(w) = 0 + 0 = 0$$
$$T(\alpha u) = \alpha T(u) = 0.$$

Therefore $\ker(T)$ is a subspace of V.

(b) Let $u, w \in R(V)$. Therefore, there exist $r, s \in V$ such that $T(r) = u, T(s) = w$. But then

$$T(r) + T(s) = T(r + s) = u + w,$$

which implies that $u + w \in R(V)$. Also $T(\alpha r) = \alpha T(r) = \alpha u$. Therefore $\alpha u \in R(V)$. Therefore $R(V)$ is a subspace.

(c) Suppose T is one-to-one. Then $T(u) = T(v)$ implies $u = v$. Suppose $k \in \ker T$. Then $T(k) = 0 = T(0)$. But T is one-to-one, so $k = 0$. Therefore $\ker(T) = \{0\}$.

Now suppose that $\ker(T) = \{0\}$. Let u, v be such that $T(u) = T(v)$. Then $T(u) - T(v) = 0$, so $T(u - v) = 0$. Therefore $u - v \in \ker T$. Therefore $u - v = 0$, so $u = v$.

(d) Suppose $\{u_1, u_2, \cdots, u_m\}$ is a basis of V. By the same strategy as shown in the proof of the previous exercise, we can show that $\{v_1, v_2, \cdots, v_k, u_{k+1}, u_{k+2}, \cdots, u_m\}$ spans V and is linearly independent, so it's a basis. We can now let $v_j = u_j$ for $j = k+1, k+2, \cdots, m$ and conclude that there is a basis of V of the form $\{v_1, v_2, \cdots, v_m\}$.
Let $x \in R(V)$. Then $T(v) = x$ for some $v \in V$. Therefore

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

But then

$$\begin{aligned} T(v) &= T((c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) + (c_{k+1} v_{k+1} + \cdots + c_m v_m)) \\ &= T(c_1 v_1 + \cdots + c_k v_k) + T(c_{k+1} v_{k+1} + \cdots + c_m v_m) \\ &= c_1 T(v_1) + \cdots + c_k T(v_k) + c_{k+1} T(v_{k+1}) + \cdots + c_m T(v_m) \\ &= c_{k+1} T(v_{k+1}) + \cdots c_m T(v_m). \end{aligned}$$

At the end we used that $T(v_1) = T(v_2) = \cdots T(v_k) = 0$ since $v_1, v_2, \ldots, v_k \in \ker(T)$. But all of this implies that x is in the span of $\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_m)\}$. To conclude we need to show that $\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_m)\}$ is linearly independent.
Suppose

$$c_{k+1} T(v_{k+1}) + c_{k+2} T(v_{k+2}) + \cdots + c_m T(v_m) = 0.$$

But since T is a linear transformation, we have

$$T(c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_m T(v_m)) = 0.$$

Therefore

$$\sum_{i=k+1}^{m} c_i v_i \in \ker(T).$$

But that means there exist $c_1, c_2, \ldots, c_k$ such that

$$\sum_{i=k+1}^{m} c_i v_i = \sum_{i=1}^{k} c_i v_i.$$

Therefore

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k + (-c_{k+1}) v_{k+1} + (-c_{k+2}) v_{k+2} + \cdots + (-c_m) v_m = 0.$$

Since $\{v_1, v_2, \cdots, v_m\}$ are linearly independent, then

$$c_1 = c_2 = \cdots = c_m = 0.$$

Therefore $\{T(v_{k+1}), T(v_{k+2}), \cdots, T(v_m)\}$ is linearly independent. Therefore it is a basis for $R(V)$ and the dimension of $R(V)$ is $m - k$.

(e) We know $m = n$. Suppose T is one-to-one, then $\ker(T) = \{0\}$. That means the dimension of $R(V)$ is $m - 0 = m = n$. Therefore $R(V)$ has the same dimension as W, therefore $R(V) = W$.
If T is onto, then $R(V) = W$. But that means the dimension of the range is $n = m$. Therefore the dimension of the null space is $m - m = 0$. That means $\ker(T) = \{0\}$, so T is one-to-one.

9. Let V and W be finite dimensional vector spaces of dimension n over a field F. Suppose that $T : V \to W$ is a vector space isomorphism. If $\{v_1, \ldots, v_n\}$ is a basis of V, show that $\{T(v_1), \ldots, T(v_n)\}$ is a basis of W. Conclude that any vector space over a field F of dimension n is isomorphic to F^n.

Solution 9. Since T is a linear transformation and it's isomorphic, so it's one-to-one, then the null space is $\{0\}$. Therefore (as in 8d), the basis for $R(V)$ is $\{T(v_1), \cdots, T(v_n)\}$. But $R(V) = W$ since T is onto.