Homework 5

Most problems below are from Judson.

1. Show that each of the following numbers is algebraic over $\mathbb{Q}$ by finding the minimal polynomial of the number over $\mathbb{Q}$.

   (a) $\sqrt{1/3 + \sqrt{7}}$

   (b) $\sqrt{3} + \sqrt[3]{5}$

   (c) $\sqrt{3} + \sqrt{2}\, i$

2. Show that each of the following numbers is algebraic over $\mathbb{Q}$ by finding the minimal polynomial of the number over $\mathbb{Q}$.

   (a) $\cos\theta + i\sin\theta$ for $\theta = 2\pi/n$ with $n \in \mathbb{N}$

   (b) $\sqrt{\sqrt[3]{2} - i}$

3. Find a basis for each of the following field extensions. What is the degree of each extension?

   (a) $\mathbb{Q}(\sqrt{3}, \sqrt{6}\,)$ over $\mathbb{Q}$

   (b) $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}\,)$ over $\mathbb{Q}$

   (c) $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$

4. Find a basis for each of the following field extensions. What is the degree of each extension?

   (a) $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7}\,)$ over $\mathbb{Q}$

   (b) $\mathbb{Q}(\sqrt{8}\,)$ over $\mathbb{Q}(\sqrt{2}\,)$

   (c) $\mathbb{Q}(\sqrt{2}, \sqrt{6} + \sqrt{10}\,)$ over $\mathbb{Q}(\sqrt{3} + \sqrt{5}\,)$

5. Determine all of the subfields of $\mathbb{Q}(\sqrt[4]{3}, i)$.

6. Show that $\mathbb{Z}_2[x]/\langle x^3 + x + 1\rangle$ is a field with eight elements. Construct a multiplication table for the multiplicative group of the field.

7. Prove or disprove: π is algebraic over $\mathbb{Q}(\pi^3)$.

8. Let $p(x)$ be a nonconstant polynomial of degree n in $F[x]$. Prove that there exists a splitting field E for $p(x)$ such that $[E : F] \leq n!$.

9. Prove or disprove: $\mathbb{Q}(\sqrt{2}\,) \cong \mathbb{Q}(\sqrt{3}\,)$.

10. Show that $\mathbb{Q}(\sqrt{3}, \sqrt{7}\,) = \mathbb{Q}(\sqrt{3} + \sqrt{7}\,)$. Extend your proof to show that $\mathbb{Q}(\sqrt{a}, \sqrt{b}\,) = \mathbb{Q}(\sqrt{a} + \sqrt{b}\,)$, where $\gcd(a, b) = 1$.