Homework 7

Most exercises are from the Judson textbook.

1. Compute each of the following Galois groups. Which of these field extensions are normal field extensions? If the extension is not normal, find a normal extension of $\mathbb{Q}$ in which the extension field is contained.
   (a) $G(\mathbb{Q}(\sqrt{30})/\mathbb{Q})$
   (b) $G(\mathbb{Q}(\sqrt[4]{5})/\mathbb{Q})$
   (c) $G(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q})$
2. Determine the Galois groups of each of the following polynomials in $\mathbb{Q}[x]$; hence, determine the solvability by radicals of each of the polynomials.
   (a) $x^5 - 12x^2 + 2$
   (b) $x^5 - 4x^4 + 2x + 2$
3. Determine the Galois groups of each of the following polynomials in $\mathbb{Q}[x]$; hence, determine the solvability by radicals of each of the polynomials.
   (a) $x^3 - 5$
   (b) $x^4 - x^2 - 6$
4. Determine the Galois groups of each of the following polynomials in $\mathbb{Q}[x]$; hence, determine the solvability by radicals of each of the polynomials.
   (a) $x^5 + 1$
   (b) $(x^2 - 2)(x^2 + 2)$
   (c) $x^8 - 1$
5. Prove that the Galois group of an irreducible quadratic polynomial is isomorphic to $\mathbb{Z}_2$.
6. Prove that the Galois group of an irreducible cubic polynomial is isomorphic to S_3 or $\mathbb{Z}_3$.
7. Let G be the Galois group of a polynomial of degree n. Prove that $|G|$ divides $n!$.
8. Construct a polynomial $f(x)$ in $\mathbb{Q}[x]$ of degree 7 that is not solvable by radicals.
9. Let $\sigma \in \text{Aut}(\mathbb{R})$. If a is a positive real number, show that $\sigma(a) > 0$.
10. Determine all of the subfields of $\mathbb{Q}(\sqrt[4]{3}, i)$.