In problem 43, we will prove

$$f(n) \geq \frac{3^n - 1}{2}.$$

Thus, we have

$$\frac{3^n + 3}{2} \leq R_n(3) \leq \lfloor en! \rfloor + 1,$$

that is,

$$3 \leq R_1(3) \leq 3, \quad 6 \leq R_2(3) \leq 6, \quad 15 \leq R_3(3) \leq 17, \quad 42 \leq R_4(3) \leq 66.$$

Because of Baumert's result, we know that even $44 \leq R_4(3) \leq 66$. The first three upper bounds are exact. The fourth is not. For about 20 years, it has been known that $R_4(3) \leq 65$, that is,

$$44 \leq R_4(3) \leq 65.$$

Problems

13. n persons meet in a room. Everyone shakes hands with everyone else. Prove that during the greeting ceremony there are always two persons who have shaken the same number of hands.

14. In a tournament with n players, everybody plays with everybody else exactly once. Prove that during the game there are always two players who have played the same number of games.

15. Twenty pairwise distinct positive integers are all < 70. Prove that among their pairwise differences there are four equal numbers.

16. Let $P_1, \ldots, P_9$ be nine lattice points in space, no three collinear. Prove that there is a lattice point L lying on some segment $P_i P_k$, $i \neq k$.

17. Fifty-one small insects are placed inside a square of side 1. Prove that at any moment there are at least three insects which can be covered by a single disk of radius 1/7.

18. Three hundred forty-two points are selected inside a cube with edge 7. Can you place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain one of the selected points?

19. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

20. S is a set of n positive integers. None of the elements of S is divisible by n. Prove that there exists a subset of S such that the sum of its elements is divisible by n.

21. Let S be a set of 25 points such that, in any 3-subset of S, there are at least two points with distance less than 1. Prove that there exists a 13-subset of S which can be covered by a disk of radius 1.

22. In any convex hexagon, there exists a diagonal which cuts off a triangle with area not more then one sixth of the hexagon.

23. If each diagonal of a convex hexagon cuts off a triangle not less than one sixth of its area, then all diagonals pass through one point, are divided by this point in the same ratio, and are parallel to the sides of the hexagon.

24. Among $n+1$ integers from $\{1, 2, \ldots, 2n\}$ there are two which are coprime.

25. From ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum (IMO 1972).

26. Let k be a positive integer and $n = 2^{k-1}$. Prove that, from $(2n-1)$ positive integers, one can select n integers, such that their sum is divisible by n.

27. Let $a_1, \cdots, a_n$ $(n \geq 5)$ be any sequence of positive integers. Prove that it is always possible to select a subsequence and add or subtract its elements such that the sum is a multiple of n^2.

28. In a room with $(m-1)n+1$ persons, there are m mutual strangers (in the room) or there is a person who is acquainted with n persons.
Does the theorem remain valid, if one person leaves the room?

29. Of k positive integers with $a_1 < a_2 < \ldots < a_k \leq n$ and $k > \lfloor (n+1)/2 \rfloor$, there is at least one pair a_i, a_r such that $a_i + a_1 = a_r$.

30. Among $(ab+1)$ mice, there is either a sequence of $(a+1)$ mice of which one is descended from the preceding, or there are $(b+1)$ mice of which none descends from the other.

31. Let a, b, c, d be integers. Show that the product of the differences $b-a, c-a, d-a, c-b, d-b, d-c$ is divisible by 12.

32. One of the positive reals $a, 2a, \ldots, (n-1)a$ has at most distance $1/n$ from a positive integer.

33. Two of six points placed into a 3×4 rectangle will have distance $\leq \sqrt{5}$.

34. In any convex $2n$-gon, there is a diagonal not parallel to any side.

35. From 52 positive integers, we can select two such that their sum or difference is divisible by 100. Is the assertion also valid for 51 positive integers?

36. Each of ten segments is longer than 1 cm but shorter than 55 cm. Prove that you can select three sides of a triangle among the segments.

37. The vertices of a regular 7-gon are colored white or black. Prove that there are vertices of the same color, which form an isosceles triangle. What about a regular 8-gon? For what regular n-gons is the assertion valid?

38. Each of nine lines partitions a square into two quadrilaterals of areas in the ratio 2:3. Then at least three of the nine lines pass through one point.

39. Among nine persons, there are three who know each other or four persons who do not know each other. The number nine cannot be replaced by a smaller one.

40. $R(4,4) = 18$ yields the problem: Among 18 persons, there are four who know each other or four persons who do not know each other. For 17 persons this need not be true.

41. $R(3,6) = 18$ gives the problem: Among 18 persons, there are three who know each other, or six who do not know each other. Try to get an estimate of $R(6,3)$ from below and above.