Friends Paradox

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In Facebook, if you average the average number of friends of everybody (which as of 2012, it was around 635 according to [2]), it is much bigger than the average number of friends of individual users (which was 190 as of 2012 according to [2]). In fact about 93% (see [2]) of people have less friends than the average number of friends their friends have. This sort of phenomenon happens in any network, not just on Facebook[1].

Let's set up the notation for our proof of this theorem. Let G be a finite graph with n vertices. For a vertex $v \in G$, we say that $u \in E(v)$ if u is a vertex of G adjacent to v (i.e., if u and v are "friends"). Let $d(v)$ be the degree of v.

The average number of "friends" is

$$\frac{1}{n}\sum_{v\in G} d(v).$$

The average of the average number of friends is

$$\frac{1}{n}\sum_{v\in G}\frac{1}{d(v)}\sum_{u\in E(v)} d(u).$$

We're ready to state the theorem (and prove it)

Theorem 1.

$$\frac{1}{n}\sum_{v\in G}\frac{1}{d(v)}\sum_{u\in E(v)} d(u) \geq \frac{1}{n}\sum_{v\in G} d(v).$$

Proof. Consider

$$A = \frac{1}{n}\sum_{v\in G}\frac{1}{d(v)}\sum_{u\in E(v)} d(u).$$

By changing the order of summation we get

$$\frac{1}{n}\sum_{u\in G} d(u)\sum_{v\in E(u)}\frac{1}{d(v)}.$$

By making a change of variable we also get

$$\frac{1}{n}\sum_{v\in G}\sum_{u\in E(v)}\frac{d(v)}{d(u)}.$$

[1] For example, on Twitter, the percentage of users with less friends than the average number of friends their friends have is over 98% according to [1] (as of 2009).

Therefore

$$\frac{1}{n}\sum_{v\in G}\frac{1}{d(v)}\sum_{u\in E(v)} d(u) = \frac{1}{n}\sum_{v\in G}\sum_{u\in E(v)}\frac{d(v)}{d(u)}.$$

Since they are equal, their average is also A. Therefore, using that $x + \frac{1}{x} \geq 2$ for all $x > 0$ (the AM-GM inequality), we get

$$\begin{aligned} A &= \frac{1}{n}\sum_{v\in G}\sum_{u\in E(v)}\frac{\frac{d(v)}{d(u)} + \frac{d(u)}{d(v)}}{2} \\ &\geq \frac{1}{n}\sum_{v\in G}\sum_{u\in E(v)} 1 \\ &\geq \frac{1}{n}\sum_{v\in G} d(v). \end{aligned}$$

Which is what we wanted to prove.

$\square$

References

[1] Nathan Oken Hodas, Farshad Kooti, and Kristina Lerman, *Friendship paradox redux: Your friends are more interesting than you*, CoRR **abs/1304.3480** (2013).

[2] Johan Ugander, Brian Karrer, Lars Backstrom, and Cameron Marlow, *The anatomy of the facebook social graph*, CoRR **abs/1111.4503** (2011).