Homework 1 Solutions

(1) 12^6

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(6) a) $10!$

b) $(10!)^3$

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(10) $X = \{a, b, \ldots, z, 0, 1, \ldots, 9\}$

$w = x_1 x_2 \cdots x_{15}$

- $x_1, x_{15} \in \{0, 1, \ldots, 9\}$ $x_1 \neq x_{15}$
- 4 symbols are t.
- 3 chars are in $\{a, e, i, o, u\}$ and distinct.

$2\binom{10}{2}$ ways of choosing x_1, x_{15}.

$\binom{13}{4}$ ways of choosing the t's.

$3!\binom{5}{3}\cdot\binom{9}{3}$ ways of choosing 3 from $\{a, e, i, o, u\}$ and the location.

The last 6 can be anything are any digits or letters different than t, a, e, i, o, u

so $\boxed{30^6}$

Therefore there are

Therefore there are

$$\boxed{2\cdot 3!\cdot\binom{10}{2}\binom{13}{4}\binom{5}{3}\binom{9}{3}(30)^6 \text{ ways}}$$

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(12) $\binom{7}{4}\binom{11}{4} = 11550$

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(15) $A+B+C+D=25$

$A, D \geq 1$

$C \leq 5$

$B \geq 4$

$A' = A-1,\ D' = D-1,\ B' = B-4$

$A' + B' + C + D' = 19$

$\binom{22}{3}$ ways but some of them have $C > 5$

so let's subtract the number of ways of $C \geq 6$, which means

$A' + B' + C' + D' = 13$

$\binom{16}{3}$ ways.

$\left(\binom{22}{3} - \binom{16}{3}\right) = 980$

$$\boxed{\binom{22}{3}-\binom{16}{3}} = 980$$

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(21) Consider the numbers
$\{1, 2, 3, \ldots, m+w\}$.

In how many ways can we choose k of them?

On the one hand $\binom{m+w}{k}$

on the other, we could restrict by how many numbers are $\leq m$.

Let j be the number of numbers $\leq m$ among the k selected from $\{1, 2, \ldots, m+w\}$.

There are $\binom{m}{j}$ ways of choosing the small numbers and $\binom{w}{k-j}$ of choosing the rest (note: if $j > m$ then $\binom{m}{j} = 0$, if $k-j > w$, then $\binom{w}{k-j} = 0$).

Therefore for each j, there are $\binom{m}{j}\binom{w}{k-j}$ ways of picking exactly j numbers $\leq m$.

Therefore $\displaystyle\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}$ $\boxed{x}$

(22) $\binom{22}{10}$

(24) $\binom{8}{3}\binom{14}{7}$

ways to get to (3,5)

ways to get from (3,5) to (10,12).

(26) $\binom{87}{14} - \binom{43}{6}\binom{44}{8}$

total paths

paths through (6,37).

(30) $\displaystyle (x^3 + 2xy^2 + y + 3)^{18} = \sum_{\substack{k_1,k_2,\\k_3,k_4}} \binom{18}{k_1,k_2,k_3,k_4} x^{3k_1}(2xy^2)^{k_2}\, y^{k_3} 3^{k_4}$

The coeff of $x^{12}y^{24}$:

$3k_1 + k_2 = 12$ $\qquad$ $k_1 + k_2 + k_3 + k_4 = 18$

$2k_2 + k_3 = 24$

Note $k_1 \leq 4$ because $3k_1 + k_2 = 12$.

$k_2 = 12 - 3k_1$ so $k_3 = 6k_1$.

Therefore $k_1 + k_2 + k_3 = 4k_1 + 12$.

since $k_1 + k_2 + k_3 \leq 18$, then $k_1 \leq 1$.

when $k_1 = 0$, $k_2 = 12$, $k_3 = 0$, $k_4 = 6$

when $k_1 = 1$, $k_2 = 9$, $k_3 = 6$, $k_4 = 2$

(note; you could also do the cases $k_1 = 2, 3, 4$ to see they yield no solutions instead of proving it by inequalities).

Therefore the answer is

$$\boxed{\binom{18}{0,12,0,6} 2^{12} 3^{6} + \binom{18}{1,9,6,2} 2^{9} \cdot 3^{2}}$$