Homework 2 Solutions

Chapter 3

(1) $r(0) = 1$, $r(1) = 26$

First letter not D: 25 choices

valid $n-1$ string: $r(n-1)$

1 C, 2 K, 7 J followed by $r(n-2)$

D followed by numbers: 10^{n-1}

Therefore

$$\boxed{r(n) = 25\,r(n-1) + 3\,r(n-2) + 10^{n-1}}$$

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(2) squares painted white and gold but no 2 cons. squares painted white.

$p(1) = 2$

W|W W|G G|G $p(2) = 3$

Suppose first square is white, next square must be gold $p(n-2)$ ways of arranging the rest.

If the first square is gold, then $p(n-1)$ ways of arranging the rest.

$$\boxed{p(n) = p(n-1) + p(n-2)}$$

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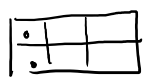
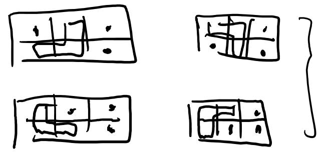
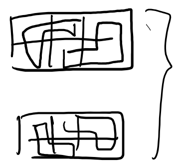
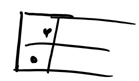
4)  $t(2) = 5$

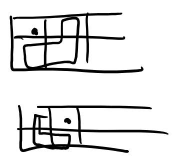
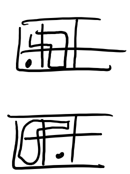
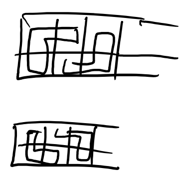
 $t(1) = 1$

 $t(2) = 5$

 4

 2 so $t(3) = 11$

For n, we can start with 

or   or 

so $t(n) = t(n-1) + 4t(n-2) + 2t(n-3)$

$$t(4) = t(3) + 4t(2) + 2t(1)$$
$$= 11 + 4(5) + 2 = \underline{33}$$

$t(5) = 33 + 4(11) + 2(5) = 87$

$t(6) = 87 + 4(33) + 2(11) = 241$

$t(7) = 241 + 4(87) + 2(33) = 655$

$$\boxed{t(7) = 655}$$

9

a) By induction: $1^2 = \frac{1(2)(3)}{6}$ ✓

Suppose that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

then

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\left[n(2n+1) + 6(n+1)\right] = \frac{n+1}{6}\left[2n^2 + 7n + 6\right]$$

$$= \frac{n+1}{6}(n+2)(2n+3) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

Therefore, the statement is true by induction. ⊠

Combinatorial:

Count the number of triples (i, j, k) satisfying $1 \leq i, j < k+1 \leq n+1$.

If $i < j$ we have to pick $i, j, k+1$ from $\{1, 2, \ldots, n+1\}$ so $\binom{n+1}{3}$ ways

If $j < i$, analogously $\binom{n+1}{3}$ ways.

If $i = j$, we need to choose $i, k+1$ from $\{1, 2, \ldots, n+1\}$ so $\binom{n+1}{2}$ ways.

Therefore $2\binom{n+1}{3} + \binom{n+1}{2}$.

On the other hand, once k is fixed, there are k^2 ways of choosing $i, j \in \{1, 2, \ldots, k\}$.

Therefore the count is $1^2 + 2^2 + \ldots + n^2$.

Therefore $1^2 + 2^2 + \cdots + n^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$

$$= \frac{2(n+1)n(n-1)}{6} + \frac{(n+1)n}{2}$$

$$= \frac{n(n+1)[2(n-1)+3]}{6} = \frac{n(n+1)(2n+1)}{6}. \quad \boxtimes$$

b) Combinatorial:

Let's count the number of ternary strings (words with "letters" from the alphabet $\{0,1,2\}$) of length n in 2 ways. On the one hand it's 3^n since each letter in the word has 3 choices.

On the other hand, we can count based on how many 2's the string has. If it has k 2's there are $\binom{n}{k}$ ways of choosing the location of the 2's and 2^{n-k} ways of choosing each letter left.

So we have

$$\binom{n}{0}2^n + \binom{n}{1}2^{n-1} + \dots + \binom{n}{n}2^0 = 3^n, \text{ but}$$

$$\binom{n}{0} = \binom{n}{n}, \quad \binom{n}{1} = \binom{n}{n-1}, \dots, \quad \binom{n}{n} = \binom{n}{0}, \text{ so}$$

$$\binom{n}{0}2^0 + \binom{n}{1}2^1 + \dots + \binom{n}{n}2^n = 3^n. \quad \boxtimes$$

INDUCTION

For $n=1$, $\binom{1}{0}2^0 + \binom{1}{1}2^1 = 1+2 = 3 = 3^1$ ✓

Suppose $\binom{n}{0}2^0 + \binom{n}{1}2^1 + \dots + \binom{n}{n}2^n = 3^n$.

Now $\binom{n+1}{0}2^0 + \binom{n+1}{1}2^1 + \dots + \binom{n+1}{n}2^n + \binom{n+1}{n+1}2^{n+1}$

$$= \binom{n}{0}2^0 + \left(\binom{n}{0}+\binom{n}{1}\right)2^1 + \left(\binom{n}{1}+\binom{n}{2}\right)2^2 + \dots + \left(\binom{n}{n-1}+\binom{n}{n}\right)2^n + \binom{n}{n}2^n$$

$$= \left(\binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \dots + \binom{n}{n}2^n\right)$$
$$+ \left(\binom{n}{0}2^1 + \binom{n}{1}2^2 + \dots + \binom{n}{n}2^{n+1}\right)$$

$$= 3^n + 2\left(\binom{n}{0}2^0 + \binom{n}{1}2^1 + \dots + \binom{n}{n}2^n\right) = 3^n + 2\cdot 3^n$$

$$= 3^n(1+2) = 3^{n+1}$$

$\boxed{X}$

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10 For $n=4$, $2^4 = 16$, $4! = 24$ $\quad 16 < 24$ ✓

Suppose $2^n < n!$

Then $2^{n+1} < 2\cdot n! \le (n+1)n!$ (for $n \ge 1$)

so $2^{n+1} < (n+1)!$

The statement is now proved by induction.

(13) Let $n=1$, $9^1-5^1=4$, so $4 \mid 9^1-5^1$.

Suppose $4 \mid 9^n-5^n$.

$$9^{n+1}-5^{n+1} = 9\cdot 9^n - 5\cdot 5^n = (5+4)\,9^n - 5\cdot 5^n$$

$$= 5\cdot 9^n - 5\cdot 5^n + 4\cdot 9^n$$

$$= 5(9^n-5^n) + 4\cdot 9^n.$$

$4 \mid 4\cdot 9^n$ and $4 \mid 9^n-5^n$ so

$$4 \mid 5(9^n-5^n) + 4\cdot 9^n = 9^{n+1}-5^{n+1} \quad \boxtimes$$

(15) For $n=1$ we have $1^3+2^3+3^3=36$.

$9 \mid 36$ so it's true for $n=1$.

Suppose $9 \mid n^3+(n+1)^3+(n+2)^3$.

Consider $(n+1)^3+(n+2)^3+(n+3)^3$

$$= (n+1)^3+(n+2)^3+(n^3+9n^2+27n+27)$$

$$= n^3+(n+1)^3+(n+2)^3 + 9(n^2+3n+3).$$

Since $9 \mid (n^3+(n+1)^3+(n+2)^3)$ and $9 \mid 9(n^2+3n+3)$, the statement follows.

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17 For $n=0$ $\quad 3n^2 - n + 2 = 2$

$$f(0) = 2 \checkmark$$

For $n=1$ $\quad 3n^2 - n + 2 = 4$

$$f(1) = 4 \checkmark$$

For $n=2$, $f(2) = 2(4) - (2) + 6 = 12$

$3n^2 - n + 2 = 12$ so it works for $n=2$.

Suppose $f(n) = 3n^2 - n + 2$.

Consider $f(n+1)$.

We have $f(n+1) = 2f(n) - f(n-1) + 6$

$$= 2(3n^2 - n + 2) - (3(n-1)^2 - (n-1) + 2) + 6$$

$$= 6n^2 - 2n + 4 - 3(n^2 - 2n + 1) + (n-1) - 2 + 6$$

$$= 3n^2 + 5n + 4 = (3n^2 + 6n + 3) - (n+1) + 2$$

$$= 3(n+1)^2 - (n+1) + 2 \quad \boxtimes$$

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19 For $n=0$, $(1+x)^0 = 1$

$1 + 0x = 1$ $\quad$ so $(1+x)^0 \geq 1 + 0x$

$\checkmark$

Suppose $(1+x)^n \geq 1+nx$.

$(1+x)^{n+1} = (1+x)(1+x)^n$. Since $x > -1$,

$1+x > 0$

so $(1+x)^{n+1} \geq (1+x)(1+nx)$

$= 1+nx+x+nx^2$

$= 1+(n+1)x+nx^2 \geq 1+(n+1)x$

since $x^2 \geq 0$.

Therefore $(1+x)^n \geq 1+nx$ for all $n \geq 0$.

$\boxtimes$