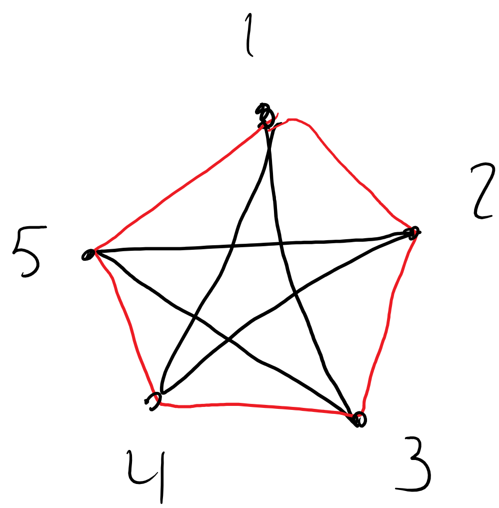
Homework 3 Solutions

③ For $X = \{1,2,3,4,5\}$

Hole 1: $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$

Hole 2: $\{\{1,4\},\{1,3\},\{2,4\},\{2,5\},\{3,5\}\}$



Hole 1 is a red edge

Hole 2 is a black edge

For $X = \{1,2,3,4,5,6\}$ it is impossible because there will be a triangle of the same color.

One can prove it as follows.

Among $\{1,2\}, \{1,3\}, \ldots, \{1,6\}$ at least 3 are in one hole. Say they are $\{1,a\}$, $\{1,b\}$ and $\{1,c\}$.

If $\{a,b\}$ is in the same hole, then $\{1,a,b\}$ has all pigeons in the same hole.

If $\{a,c\}$ then $\{1,a,c\}$.

If $\{b,c\}$ then $\{1,b,c\}$.

So $\{a,b\}, \{a,c\}, \{b,c\}$ would all have to be in the other hole, but then $\{a,b,c\}$ would have all pigeons in the same hole.

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13 If someone shakes no hands, then no one shakes $n-1$ hands and viceversa.

Therefore either you pick n among

$0, 1, 2, \dots, n-2$ or

n among $1, 2, \dots, n-1$

In either case you have n people and $n-1$ possibilities, so two must have the same number of handshakes.

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16 If (a,b,c) and (d,e,f) satisfy

a, d have the same parity
b, e have the same parity
c, f have the same parity

Then the midpoint $\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$ is a lattice point

There are $2^3 = 8$ possible configurations

(even, even, even), (even, even, odd), ..., (odd, odd, odd)

By pigeonhole 2 of the 9 points have the same parity in the three coordinates.

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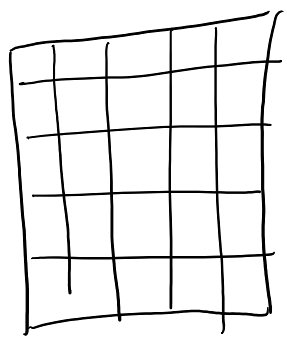
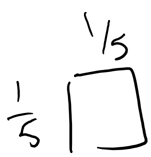
(17)

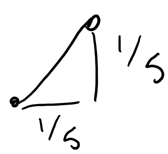


Divide the 1x1 square in 25 $\frac{1}{5} \times \frac{1}{5}$ squares.

By pigeonhole 3 of the insects must land in one of these.



The farthest apart they can be is



$$\sqrt{\frac{1}{5^2} + \frac{1}{5^2}} = \frac{\sqrt{2}}{5}.$$

Place the circle at the center of the square

the radius is $\frac{\sqrt{2}}{10}$.

$$\left(\frac{\sqrt{2}}{10}\right)^2 = \frac{2}{100} = \frac{1}{50} < \frac{1}{49} \qquad 50$$

$\frac{\sqrt{2}}{10} < \frac{1}{7}$. Therefore

3 insects are contained in a circle of radius $\frac{1}{7}$.

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(19) Let $a_1 = 1$, $a_2 = 11$, $a_3 = 111$, ...

Given $n+1$ of these, by pigeonhole two would have the same remainder mod n.

Say $a_i \equiv a_j \mod n$

$$a_i = 111 \cdots 1$$
$$a_j = 11 \cdots 1$$

$$a_i - a_j = \underbrace{111 \cdots 1}_{i-j}\underbrace{00 \cdots 0}_{j}$$ is a multiple of n.

Since 2, 5 don't divide n, then

$$n \mid \underbrace{11 \cdots 1}_{i-j}.$$ ⊠

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(25) The number of subsets is $2^{10} = 1024$.

The maximal sum is $100 + 99 + \ldots + 91 \leq 1000$.

Therefore, by Pigeonhole 2 distinct subsets must have the same sum. They might not be disjoint. We can remove their intersection. What's left is now two disjoint subsets, since you removed the same from both sets, the sums are still the same.

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(32) Consider $\{a\}, \{2a\}, \ldots, \{(n-1)a\}$ (fractional parts). If any is in the intervals $[0, \frac{1}{n}]$ or $[\frac{n-1}{n}, 1]$ then we're done. Suppose that all of them land in $[\frac{1}{n}, \frac{n-1}{n}]$. Then

Consider the $n-2$ subintervals

$$\left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \dots, \left[\frac{n-2}{n}, \frac{n-1}{n}\right]$$

By pigeonhole two fractional parts must land in the same subinterval.

Then $|\{ia\} - \{ja\}| \leq \frac{1}{n}$.

Since

$$|\{ia\} - \{ja\}| = |\{(i-j)a\}|,$$

$$|\{(i-j)a\}| \leq \frac{1}{n} \quad \text{but} \quad i-j \in \{1, 2, \dots, n-1\}$$

which is what we wanted to prove.

35

$$\left.\begin{array}{c} \{0\} \\ \{1, 99\} \\ \vdots \\ \{49, 51\} \\ \{50\} \end{array}\right] \quad 51 \text{ sets.}$$

Consider the positive integers mod 100 . 1/

By pigeonhole 2 numbers must fall in one of those.

If $a \equiv b \bmod 100$ then $100 | a-b$.

If $a \not\equiv b \bmod 100$, then $100 | a+b$ by construction of the sets. $\boxtimes$