Homework 4 Solutions

1. a) 2

   b) 4

   c) 5, $\{1, 6, 8, 9, 4\}$

   d) $(1, 5, 2, 3, 8, 9, 10, 7, 1)$

   e) 2

   f) 3

   g) $(4, 7, 1, 5, 2, 6)$

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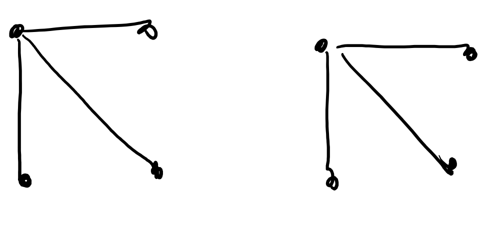
2. 

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(4) For a division to have 5 games within the division for all 7 teams, you'd need $\frac{5 \times 7}{2}$ games. But $\frac{35}{2}$ is not an integer. Therefore, it's impossible.

ALTERNATIVE SOLUTION

In terms of graph theory, the induced subgraph on one of the pools has 7 vertices and every vertex has degree 5. But then $\sum^{7} \deg(v) = 35$ is odd. !?

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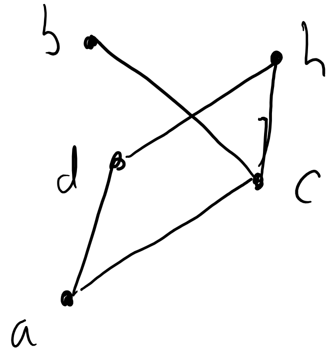
(5) a) Yes.

b) No. cj is not an edge.

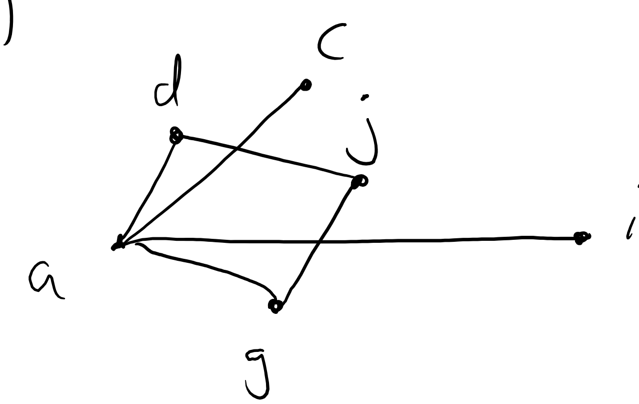
c) The induced subgraph is

The induced subgraph is

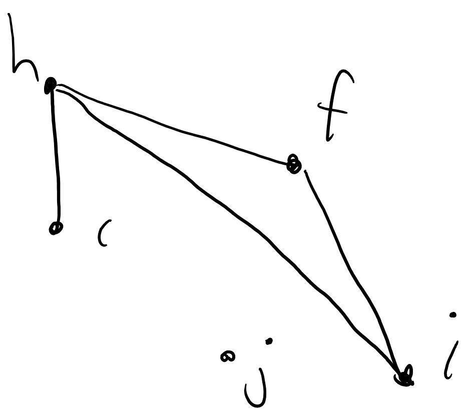


dh is missing from E_3, so (V_3, E_3) is not an induced subgraph.

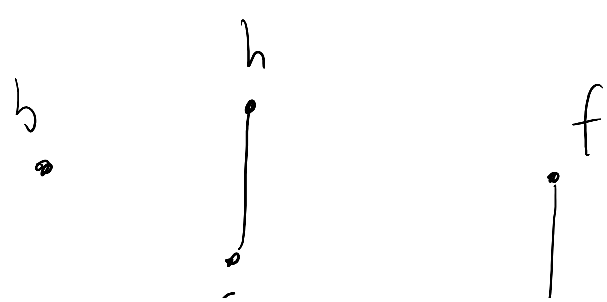
d)



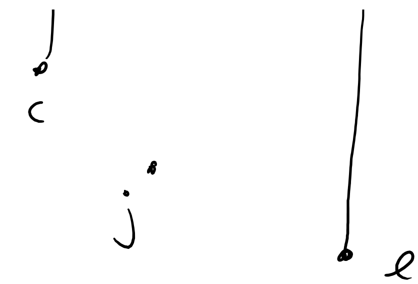
e)



f)

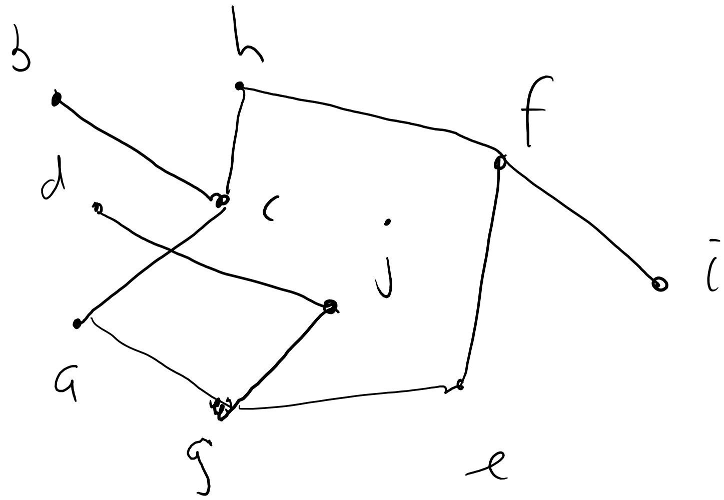


Many others exist.



exist.

g)



(You can choose any 10 edges of the original as long as you use all vertices.)

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7. G_3 is not isomorphic to G_2 as G_3 contains K_3 but G_2 does not.

G_4 is not isomorphic to anything as it has a vertex of degree $\underline{1}$ (x_2) and no other graph does.

$G_1 \cong G_3$ (so $G_1 \ncong G_2$)

$w_1 \mapsto v_1$ $w_2 \mapsto v_5$ $w_3 \mapsto v_6$

$w_4 \mapsto v_3$ $w_5 \mapsto v_2$ $w_6 \mapsto v_4$

$w_1 w_2 \longleftrightarrow v_1 v_5$ ✓

$w_2 w_3 \longleftrightarrow v_5 v_6$ ✓

$w_3 w_4 \longleftrightarrow v_6 v_3$ ✓

$w_4 w_5 \longleftrightarrow v_3 v_2$ ✓

$w_5 w_6 \longleftrightarrow v_2 v_4$ ✓

$w_3 w_5 \longleftrightarrow v_6 v_2$ ✓

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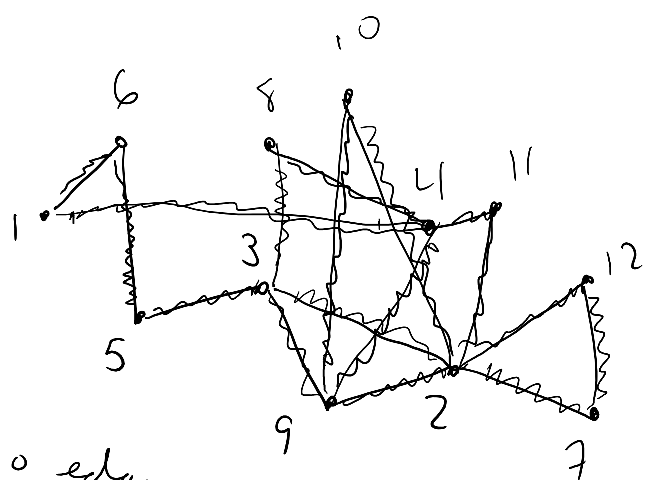
(8) (1)



(1, 4, 8, 3, 2, 7, 12, 2, 9, 3, 5, 6, 1)

4 has not visited two edges

$\boxed{(1, 4, 9, 10, 2, 11, 4, 8, 3, 2, 7, 12, 2, 9, 3, 5, 6, 1)}$

It is Eulerian

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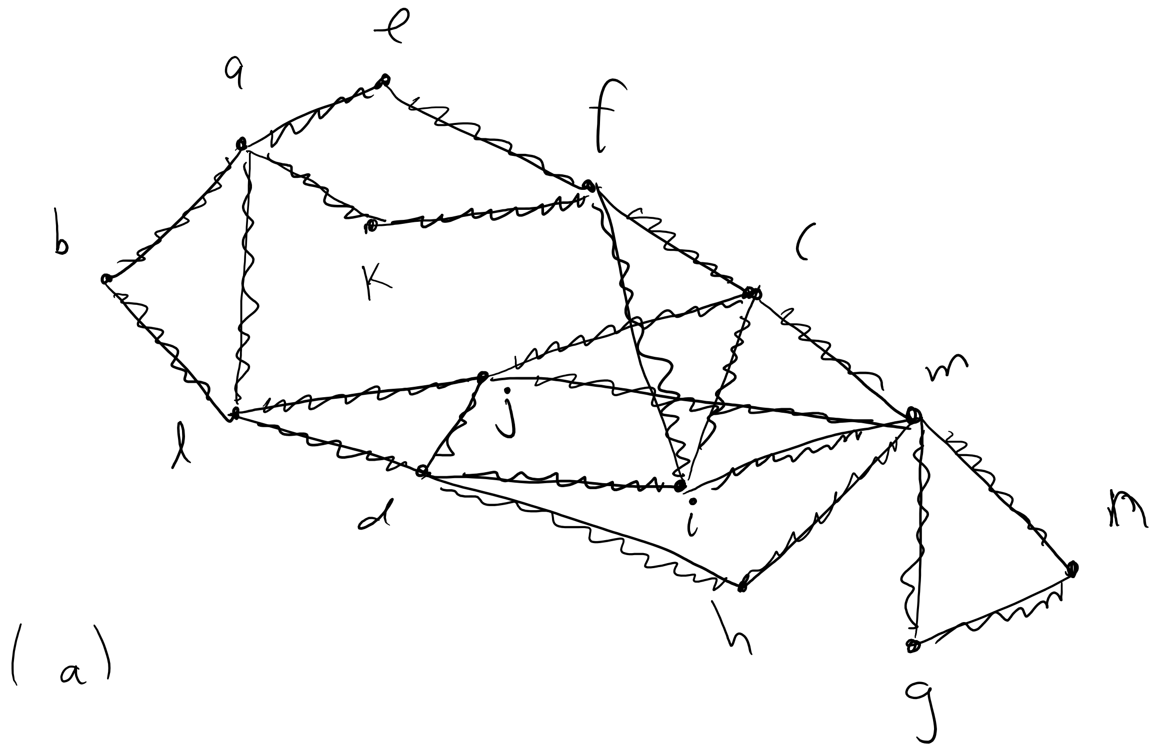
(9) Every vertex has even degree, so it's Eulerian.



(a)

(a, b, l, a)

$\boxed{(a, e, f, c, i, d, j, c, m, n, g, m, h, d, l, j, m, i, f, k, a, b, l, a)}$

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(10) vertex 5 and vertex 12 have odd degree, so it's not Eulerian.

By adding the edge $5 \leftrightarrow 12$, every vertex has even degree, so it's Eulerian.

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(11) Suppose there is an Eulerian trail in graph G.
Then there is a path connecting any 2 vertices since $(x_0, \cancel{x_1}, \ldots, x_t)$ contains all vertices of G.

So the two vertices are x_i, x_j for some $i<j$.
Then they are path connected $x_i x_{i+1}, x_{i+1} x_{i+2}, \ldots, x_{j-1} x_j$.

Therefore G is connected.

Let $v \in V$ be such that $v \neq x_0, v \neq x_t$.

Then $v = x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ for some
$1 \leq i_1 < i_2 < \ldots < i_k \leq t-1$ (with k possibly 1)

Note $x_{i_1} x_{i_1+1}$ is an edge so $i_1+1 \neq i_2$.
Similarly $i_m+1 \neq i_{m+1}$.

Now the edges that include v are
$x_{i_1-1} x_{i_1}, x_{i_1} x_{i_1+1}, x_{i_2-1} x_{i_2}, x_{i_2} x_{i_2+1}, \ldots, x_{i_k-1} x_{i_k}, x_{i_k} x_{i_k+1}$

so there are $2k$ edges. So the degree of any vertex v besides x_0, x_t is even. Therefore it has at most 2 vertices of odd degree (x_0, x_t).

Now suppose G is connected and has at most

2 vertices of odd degree. If it has 0 vertices of odd degree, then by the theorem proved in class, there is an Eulerian circuit (and hence a trail).
If it had 1 odd degree vertex

$$\sum_{v \in G} \deg(v) \text{ is odd, but that}$$

is impossible since $\sum_{v \in G} \deg(v) = 2|E|$ is even.

Therefore, we may assume there are exactly 2 vertices of odd degree. Let's call them v, w. Let's consider the following cases

Case 1: vw is not an edge.

Then if we add that edge, we can form an Eulerian circuit $(v, x_1, x_2, \dots, x_{t-1}, x_t, v)$ in this new graph because every vertex has even degree.
Since it is a circuit we can force wv to be at the end, i.e. $(v, y_1, y_2, \dots, y_s, w, v)$ (after reordering).
But then $(v, y_1, \dots, y_s, w)$ is an Eulerian trail in G.

Case 2: vw is an edge in G. Let G' be the graph with vw removed. If G' is disconnected consider the connected component A containing v and the connected component B containing w. Each subgraph has all vertices of even degree, so they have Eulerian circuits

$(v, x_1, x_2, \ldots, x_t)$ with $v = x_t$

and $(w, y_1, y_2, \ldots, y_s)$ with $w = y_s$.

Then in the original graph

$(v, x_1, \ldots, x_t, w, y_1, y_2, \ldots, y_s)$ is an

Eulerian trail.

If G' is not disconnected, then all vertices have even degree, so there is an Eulerian circuit $(v, x_1, x_2, \ldots, x_t)$ with $x_t = v$.

Then $(v, x_1, x_2, \ldots, x_t, w)$ is an Eulerian trail in G. $\boxtimes$

(12) There are Hamiltonian graphs with 17 vertices and 129 edges, for example add 112 edges to C_{17}.

Since $\binom{17}{2} - 17 = 119$, you can find 112 edges to add.

$|K_{17}| = \binom{17}{2} = \frac{17 \cdot 16}{2} = 17 \cdot 8 = 136$.

So 129 edges is 7 less than 136. Therefore, every vertex has degree $\geq 16 - 7 = 9 = \lceil \frac{17}{2} \rceil$.

Therefore the graph has to be Hamiltonian

Bob is right