Homework 7 Solutions

(2) 1189 students

C++	856	C++ & Java	639
Java	792	C++ & Python	519
Python	692	Java & Python	632

C++ & Java & Python 488

$$|\text{C++ or Java or Python}| = 856 + 792 + 692 - 639 - 519 - 632 + 488 = 1038.$$

Therefore there are $1189 - 1038 = 151$ students who reported not knowing any of the three languages.

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(5)

$$1000 - \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{8} \right\rfloor - \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{24} \right\rfloor + \left\lfloor \frac{1000}{75} \right\rfloor + \left\lfloor \frac{1000}{200} \right\rfloor - \left\lfloor \frac{1000}{600} \right\rfloor$$

$$= 1000 - 333 - 125 - 40 + 41 + 13 + 5 - 1$$

$$= \underline{560}$$

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⑥ $f+g+d+c_1+c_2=173$

$f,g,d,c_1,c_2\geq 1$

$c_2\leq 10$

$c_1\leq 30$

$f'+g'+d'+c_1'+c_2'=168 \qquad c_1'\leq 29,\ c_2'\leq 9$

No restriction on c_1', c_2': $\binom{172}{4}$

remove $c_1'\geq 30 \qquad -\binom{142}{4}$

remove $c_2'\geq 10 \qquad -\binom{162}{4}$

put in $c_1'\geq 30,\ c_2'\geq 10 \qquad +\binom{132}{4}$.

Therefore $\underline{\binom{172}{4}-\binom{142}{4}-\binom{162}{4}+\binom{132}{4}=3409975}$

If we add the restriction that $f\geq 5$, then

$$\binom{168}{4}-\binom{138}{4}-\binom{158}{4}+\binom{128}{4}=3231775$$

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⑨ Eats lunch 15 times. Let a be the number of times he eats alone.

$$15 = a+6\cdot 11-\binom{6}{2}\cdot 9+\binom{6}{3}\cdot 6-\binom{6}{4}\cdot 4+\binom{6}{5}\cdot 4-\underline{1}$$

$$15 = a+66-135+120-60+24-\underline{1}=a+14.$$

He ate one alone.

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(11) a) It does not satisfy P_2 since $f(2)=2$ (and $f(6)=2$, $f(8)=2$).

It does satisfy P_3 since 3 is not in the image.

Satisfies P_3, P_5, P_7

b) Yes. $g(1)=1, g(2)=2, \ldots, g(7)=7, g(8)=1$.

c) No because the function can't be onto when $m>n$ and $9>8$.

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(14) $\{1,\ldots,8\} \to \{1,2,\ldots,6\}$

$$6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}1^8 = 191520$$

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(21) There are $\binom{7}{3}$ ways of picking the three clerks that get the right envelope.

Now we want to place the other 4 so that no one gets theirs. This would be the number of derangements when $n=4$, $4!\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$

So $\binom{7}{3}4!\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)=\frac{7!}{3!}\left(\frac{3}{8}\right)=\frac{7!}{16}=315.$

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(22) a) Let k be such that $\sigma(k)=1$.

If $\sigma(1)=k$ then we have a permutation from $\{1,\dots,n\}\setminus\{1,k\}$ to $\{1,\dots,n\}\setminus\{1,k\}$ where no element is fixed, so it's a derangement and there are d_{n-2} ways of arranging this.

Since k has $n-1$ possibilities, we have

$(n-1)d_{n-2}$ ways of this happening.

If $\sigma(1)\neq k$, then what's left of σ is a derangement on $\{1,2,\dots,k-1,k+1,\dots,n\}$

So there are d_{n-1} ways. Since there are $n-1$ choices for k, we have $(n-1)d_{n-1}$.

Combining both cases we see

$$\boxed{d_n=(n-1)(d_{n-1}+d_{n-2})}$$

b) We want to show $d_n=nd_{n-1}+(-1)^n$ for $n\geq 2$

Lets prove it by induction.

We know $d_1 = 0,\ d_2 = 1,\ d_3 = (2)(d_1 + d_2) = 2$

$$d_4 = 3(d_2 + d_3) = 9$$

$n d_{n-1} + (-1)^n$: when $n = 2$ we get $2d_1 + (-1)^2 = 1$ ✓

when $n = 3$ we get $3d_2 + (-1)^3 = 2$ ✓

when $n = 4$ we get $4d_3 + (-1)^4 = 8 + 1 = 9$ ✓

So we have the base cases.

Suppose $d_n = n d_{n-1} + (-1)^n$,

because $-(-1)^n = (-1)^{n+1}$

so $\quad d_n + (-1)^{n+1} = n d_{n-1}$

Now, by (a), we have

$$d_{n+1} = (n+1-1)(d_n + d_{n-1})$$

$$= n d_n + n d_{n-1}$$

$$= n d_n + d_n + (-1)^{n+1}$$

$$= (n+1) d_n + (-1)^{n+1} \quad \boxed{\times}$$

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(24)

756	2
378	2
189	3
63	3

$$756 = 2^2 \cdot 3^3 \cdot 7$$

$7 \mid 7$

1

$\phi(756) = 756(1 - \frac{1}{2})(1 - \ldots$

$$= 2^2 \cdot 3^3 \cdot 7 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) \cdot \left(\frac{6}{7}\right) = 2^3 \cdot 3^3 = \underline{216}$$

$180 = 2^2 \cdot 3^2 \cdot 5$

(25) $\phi(1625190883965792)$

$$= 2^5 \cdot 3^4 \cdot 11^2 \cdot 13 \cdot 23^3 \cdot 181^2 \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{10}{11}\right)\left(\frac{12}{13}\right)\left(\frac{22}{23}\right)\left(\frac{180}{181}\right)$$

$$= 2^4 \cdot 3^3 \cdot 11 \cdot 23^2 \cdot 181 \cdot 2 \cdot 2 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \cdot 11 \cdot 2^2 \cdot 3^2 \cdot 5$$

$$= 2^{11} \cdot 3^6 \cdot 5^2 \cdot 11^2 \cdot 23^2 \cdot 181$$

$$= 432431285299200$$