Homework 1 Solutions

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1 Chapter 1

Problem 1. (**Exercise 4**)
Prove $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Solution 1.

$$A \cup \emptyset = \{x \in A \text{ or } x \in \emptyset\} = \{x \in A\} = A$$

and

$$A \cap \emptyset = \{x \in A \text{ and } x \in \emptyset\} = \emptyset.$$

If proving it this way seems unsatisfactory, an **alternative proof** of the first statement would be:
Let $x \in A \cup \emptyset$, then $x \in A$ or $x \in \emptyset$, but since $x \notin \emptyset$, then $x \in A$, so $A \cup \emptyset \subseteq A$.
Now, if $x \in A$, then $x \in A \cup \emptyset$, so $A \subseteq A \cup \emptyset$. Therefore $A \cup \emptyset = A$.
An **alternative proof** for the second statement can be proved as follows:
For the sake of contradiction, suppose $A \cap \emptyset \neq \emptyset$. Then there exists an $x \in A \cap \emptyset$. But then $x \in A$ and $x \in \emptyset$. But there is no $x \in \emptyset$, therefore we've reached a contradiction. Hence $A \cap \emptyset = \emptyset$.

Problem 2. (**Exercise 13**)
Prove $(A \cup B) \setminus B = A \setminus B$.

Solution 2. Let's first prove that $(A \cup B) \setminus B \subseteq A \setminus B$.
Suppose $x \in (A \cup B) \setminus B$. Then $x \in (A \cup B)$ and $x \notin B$. Since $x \in (A \cup B)$, $x \in A$ or $x \in B$. But $x \notin B$, which forces $x \in A$. Therefore $x \in A$ and $x \notin B$, which implies that $x \in A \setminus B$.
Now let's prove that $A \setminus B \subseteq (A \cup B) \setminus B$.
Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. Since $x \in A$, then $x \in (A \cup B)$. Therefore $x \in (A \cup B)$ and $x \notin B$, which implies $x \in (A \cup B) \setminus B$.

Problem 3. (**Exercise 18**)
Determine which of the following functions are one-to-one and which are onto. If the function is not onto, determine its range.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + 3$

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$

(d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$

Solution 3.

(a) f is one-to-one because if $e^x = e^y$, then $e^{x-y} = 1$, which implies that $x - y = 0$ and hence $x = y$.
f is not onto because e^x is never negative. The range of f is $\{x \in \mathbb{R} \mid x > 0\}$.

(b) f is not one-to-one because $f(1) = f(-1) = 4$.
f is not onto because $n^2 + 3$ is never negative. The range of f is $\{n^2 + 3 \mid n \in \mathbb{Z}\}$.

(c) f is not one-to-one because $\sin 0 = \sin \pi = 0$.
f is not onto because $|\sin x| \leq 1$. The range of f is $\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$.

(d) f is not one-to-one because $f(1) = f(-1) = 1$.
f is no onto because f is never negative. The range of f is $\{x^2 \mid x \in \mathbb{Z}\}$.

Problem 4. (**Exercise 22**)
Let $f : A \to B$ and $g : B \to C$ be maps.

(a) If f and g are both one-to-one functions, show that $g \circ f$ is one-to-one.

(b) If $g \circ f$ is onto, show that g is onto.

(c) If $g \circ f$ is one-to-one, show that f is one-to-one.

(d) If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.

(e) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

Solution 4.

(a) Suppose $g \circ f(x) = g \circ f(y)$, i.e., $g(f(x)) = g(f(y))$. Since g is one-to-one, then $f(x) = f(y)$. Since f is one-to-one, then $x = y$. Therefore $g \circ f$ is one-to-one.

(b)

(c) Suppose $f(x) = f(y)$, then $g(f(x)) = g(f(y))$. Since $g \circ f$ is one-to-one and $g \circ f(x) = g \circ f(y)$, then $x = y$. Therefore f is one-to-one.
An **alternative proof** would be to assume for the sake of contradiction that f is not one-to-one, i.e., there exist distinct x and y such that $f(x) = f(y)$. But then $g(f(x)) = g(f(y)$, which implies that $x = y$, contradicting the fact that x and y are distinct.
The two proofs are very similar but I wrote both of them to illustrate that you don't have to think about it a certain way.

(d)

(e) Let $b \in B$. Now consider $c = g(b) \in C$. Since $g \circ f$ is onto, then there exists an $a \in A$ such that $g \circ f(a) = c$. Therefore $g(f(a)) = c = g(b)$. Since g is one-to-one, $f(a) = b$. So we've shown that there exists an $a \in A$ such that $f(a) = b$, which shows that f is onto.

Problem 5. (**Exercise 24**)
Let $f : X \to Y$ be a map with $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$.

(a) Prove $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(b) Prove $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Give an example in which equality fails.

(c) Prove $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$, where

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

(d) Prove $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(e) Prove $f^{-1}(Y \setminus B_1) = X \setminus f^{-1}(B_1)$.

Solution 5.

(a) Let $x \in f(A_1 \cup A_2)$. Then there exists a $y \in A_1 \cup A_2$ such that $f(y) = x$. Since $y \in A_1 \cup A_2$, $y \in A_1$ or $y \in A_2$, so $x \in f(A_1)$ or $x \in f(A_2)$, so $x \in f(A_1) \cup f(A_2)$.
Now, let $x \in f(A_1) \cup f(A_2)$, so $x \in f(A_1)$ or $x \in f(A_2)$. If $x \in f(A_1)$, then there exists $y \in A_1$ such that $f(y) = x$. Since $y \in A_1$, then $y \in A_1 \cup A_2$, so $x \in f(A_1 \cup A_2)$. Similarly, if $x \in f(A_2)$, then there exists $y \in A_2$ such that $f(y) = x$. Since $y \in A_2$, then $y \in A_1 \cup A_2$, so $x \in f(A_1 \cup A_2)$. In either case, $x \in f(A_1 \cup A_2)$. Therefore $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(b) Let $x \in f(A_1 \cap A_2)$. Then there exists $y \in A_1 \cap A_2$ such that $f(y) = x$. Since $y \in A_1$, then $x \in f(A_1)$. Similarly, since $y \in A_2$, we have that $x \in f(A_2)$. Since $x \in f(A_1)$ and $x \in f(A_2)$, we can conclude that $x \in f(A_1) \cap f(A_2)$, which proves that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

An example where equality fails is the following. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Now, let $A_1 = \{-1, 2\}$. Let $A_2 = \{1, 2\}$. So $f(A_1 \cap A_2) = f(\{2\}) = \{4\}$, while $f(A_1) \cap f(A_2) = f(\{-1, 2\}) \cap f(\{1, 2\}) = \{1, 4\} \cap \{1, 4\} = \{1, 4\} \neq \{4\}$.

(c) Let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$. So $f(x) \in B_1$ or $f(x) \in B_2$. Therefore $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, which implies that $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Now, let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Therefore $f(x) \in B_1$ or $f(x) \in B_2$, which implies that $x \in f^{-1}(B_1 \cup B_2)$.

(d) Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$. So $f(x) \in B_1$ and $f(x) \in B_2$. Therefore $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, which implies that $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

Now, let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Therefore $f(x) \in B_1$ and $f(x) \in B_2$, which implies that $x \in f^{-1}(B_1 \cap B_2)$.

(e) Let $x \in f^{-1}(Y \setminus B_1)$. Then $f(x) \in Y \setminus B_1$, so $f(x) \in Y$ and $f(x) \notin B_1$. Since $f(x) \in Y$, then $x \in X$. Since $f(x) \notin B_1$, then $x \notin f^{-1}(B_1)$. Therefore $x \in X \setminus f^{-1}(B_1)$.

Now, let $x \in X \setminus f^{-1}(B_1)$. Then $x \in X$ and $x \notin f^{-1}(B_1)$. Since $x \in X$, $f(x) \in Y$. Since $x \notin f^{-1}(B_1)$, $f(x) \notin B_1$. Therefore $f(x) \in Y \setminus B_1$, which implies that $x \in f^{-1}(Y \setminus B_1)$.

Problem 6. (**Exercise 25**)

Determine whether or not the following relations are equivalence relations on the given set. If the relation is an equivalence relation, describe the partition given by it. If the relation is not an equivalence relation, state why it fails to be one.

(a) $x \sim y$ in $\mathbb{R}$ if $x \geq y$

(b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$

(c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leq 4$

(d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod 6$

Solution 6.

(a) It's not an equivalence relation because it is not symmetric. For example $3 \sim 2$ because $3 \geq 2$, but $2 \not\sim 3$ since $2 \ngeq 3$.

(b) It's not an equivalence relation because it is not reflexive since $0 \not\sim 0$.

(c) It's not an equivalence relation because it is not transitive. Indeed, $4 \sim 0$ since $|4 - 0| \leq 4$ and $0 \sim -4$ since $|0 - (-4)| \leq 4$. Yet $4 \not\sim -4$ because $|4 - (-4)| = 8 > 4$.

(d) It is an equivalence relation. The equivalence relation partitions the set $\mathbb{Z}$ into the following six equivalence classes:

- $[0]$ is the set of multiples of 6.
- $[1]$ is the set of numbers of the form $6k + 1$ for some integer k.
- $[2]$ is the set of numbers of the form $6k + 2$ for some integer k.
- $[3]$ is the set of numbers of the form $6k + 3$ for some integer k.
- $[4]$ is the set of numbers of the form $6k + 4$ for some integer k.
- $[5]$ is the set of numbers of the form $6k + 5$ for some integer k.

2 Chapter 2

Problem 7. (**Exercise 1**)
Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for $n \in \mathbb{N}$.

Solution 7. We will prove it by induction.
The statement is true for $n = 1$ since, the left hand side of the equation is $1^2 = 1$ and the right hand side is

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1.$$

Assume that for an integer $k \geq 1$, the statement is true for $n = k$, i.e., we have that

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Therefore

$$\begin{aligned}
1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= (1^2 + 2^2 + \ldots + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
&= (k+1)\left(\frac{2k^2 + k + 6(k+1)}{6}\right) \\
&= \frac{k+1}{6}\left(2k^2 + 7k + 6\right) \\
&= \frac{k+1}{6}\left((k+2)(2k+3)\right) \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

Therefore the statement is true for $n = (k+1)$. The proof is complete.

Problem 8. (**Exercise 3**)
Prove that $n! > 2^n$ for $n \geq 4$.

Solution 8.

Problem 9. (**Exercise 7**)
Show that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k,$$

for $n \in \mathbb{N}$ and positive real numbers $a_1, a_2, \ldots a_n$.

Solution 9. First note that it is true for $n = 1$ since in that case $\sqrt[1]{a_1} = a_1 = \frac{a_1}{1}$.
We will first prove it for the powers of 2, so let $n = 2^r$. We will do induction on r (the exponent of 2).
When $r = 1$, we want to prove

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

By squaring both sides we see that we want to prove

$$\left(\frac{a_1 + a_2}{2}\right)^2 \geq a_1 a_2.$$

(Note: Squaring is not always valid when proving inequalities. In this case it is valid because the a_i are positive.)
So we want to prove that

$$\begin{aligned} a_1^2 + 2a_1a_2 + a_2^2 &\geq 4a_1a_2 \\ a_1^2 - 2a_1a_2 + a_2^2 &\geq 0 \\ (a_1 - a_2)^2 &\geq 0. \end{aligned}$$

But the last statement is clearly true, since $x^2 \geq 0$ for any $x \in \mathbb{R}$.

So we've proven the statement for $r = 1$, i.e., for $n = 2^1 = 2$. Now, to continue by induction suppose that the inequality is true for $r = k$, i.e.,

$$\frac{1}{2^k}\sum_{i=1}^{2^k} a_i \geq \sqrt[2^k]{a_1a_2a_3\cdots a_{2^k}}.$$

Now we want to prove it for $r = k+1$, i.e., we want to prove

$$\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} a_i \geq \sqrt[2^{k+1}]{a_1a_2a_3\cdots a_{2^{k+1}}}.$$

Let's manipulate the left hand side by first expanding the sum, then grouping in pairs and using the inequality for $n = 2$ and finally using the induction hypothesis (the case $n = 2^k$):

$$\begin{aligned} \frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} a_i &= \frac{a_1 + a_2 + a_3 + \ldots + a_{2^{k+1}-1} + a_{2^{k+1}}}{2^{k+1}} \\ &= \frac{\left(\frac{a_1+a_2}{2}\right) + \left(\frac{a_3+a_4}{2}\right) + \ldots + \left(\frac{a_{2^{k+1}-1}+a_{2^{k+1}}}{2}\right)}{2^k} \\ &\geq \frac{\sqrt{a_1a_2} + \sqrt{a_3a_4} + \ldots + \sqrt{a_{2^{k+1}-1}a_{2^{k+1}}}}{2^k} \\ &\geq \sqrt[2^k]{\sqrt{a_1a_2}\sqrt{a_3a_4}\cdots\sqrt{a_{2^{k+1}-1}a_{2^{k+1}}}} \\ &= \sqrt[2^k]{\sqrt{a_1a_2a_3a_4\cdots a_{2^{k+1}-1}a_{2^{k+1}}}} \\ &= \sqrt[2^{k+1}]{a_1a_2a_3a_4\cdots a_{2^{k+1}-1}a_{2^{k+1}}}. \end{aligned}$$

So we have proven the inequality for all powers of 2. So now we know the statement is true for $n = 1, 2, 4, 8, 16, 32, 64, 128, \ldots$, but we don't know what happens with the values in between. To prove the values in between we'll use "reverse-induction". "Reverse-induction" is where you prove that if the inequality is true for $n = k$, then the inequality is true for $n = k - 1$. If we manege to prove this implication, we can fill in the holes. For example, since we know the inequality for powers of 2, it is true for $n = 64$, if we can prove that $k \to k - 1$, then it would be true for $n = 63$, but then it would be true for $n = 62$, and so on. The reason reverse-induction is a valid strategy is that we already proved it for all powers of 2, so we can leap to the next power of 2 and then fill in the gaps below it.

Without further ado, let's prove the "reverse-induction" implication. Suppose the inequality is true for $n = k$, i.e.,

$$\frac{a_1 + a_2 + \ldots + a_k}{k} \geq \sqrt[k]{a_1a_2\cdots a_k}.$$

Now let's prove it for $n = k-1$. Suppose we have $b_1, b_2, \ldots b_{k-1}$ be positive real numbers. We want to show

$$\frac{b_1 + b_2 + \ldots + b_{k-1}}{k-1} \geq \sqrt[k-1]{b_1b_2\cdots b_{k-1}}.$$

Let

$$b_k = \frac{b_1 + b_2 + \ldots + b_{k-1}}{k-1}.$$

By the "reverse-induction" hypothesis, we know

$$\frac{b_1 + b_2 + \ldots + b_{k-1} + b_k}{k} \geq \sqrt[k]{b_1 b_2 \cdots b_{k-1} b_k}. \tag{1}$$

The left-hand-side is:

$$\begin{aligned}\frac{b_1 + b_2 + \ldots + b_{k-1} + b_k}{k} &= \frac{b_1 + b_2 + \ldots + b_{k-1} + \left(\frac{b_1 + b_2 + \ldots + b_{k-1}}{k-1}\right)}{k} \\ &= \frac{(k-1)(b_1 + b_2 + \ldots + b_{k-1}) + (b_1 + b_2 + \ldots + b_{k-1})}{k(k-1)} \\ &= \frac{b_1 + b_2 + \ldots + b_{k-1}}{k-1}.\end{aligned}$$

And the right-hand-side is:

$$\begin{aligned}\sqrt[k]{b_1 b_2 \cdots b_{k-1} b_k} &= \sqrt[k]{b_1 b_2 \cdots b_{k-1}\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)} \\ &= \sqrt[k]{b_1 b_2 \cdots b_{k-1}}\sqrt[k]{\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)} \\ &= (b_1 b_2 \cdots b_{k-1})^{\frac{1}{k}}\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)^{\frac{1}{k}}.\end{aligned}$$

Therefore (1) becomes

$$\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1} \geq (b_1 b_2 \cdots b_{k-1})^{\frac{1}{k}}\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)^{\frac{1}{k}},$$

so

$$\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)^{1-\frac{1}{k}} \geq (b_1 b_2 \cdots b_{k-1})^{\frac{1}{k}},$$

which implies (by raising both sides to the k) that

$$\left(\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1}\right)^{k-1} \geq (b_1 b_2 \cdots b_{k-1}).$$

Now by taking the $(k-1)$-th root on both sides (valid because both sides are positive), we get the desired inequality, i.e.,

$$\frac{b_1 + b_2 + \ldots b_{k-1}}{k-1} \geq \sqrt[k-1]{b_1 b_2 \cdots b_{k-1}}.$$

The "reverse-induction" is complete and hence the statement is true for all $n \in \mathbb{N}$.

Problem 10. (**Exercise 15**)
For each of the following pairs of numbers a and b, calculate $\gcd(a, b)$ and find integers r and s such that $\gcd(a, b) = ra + sb$.

(a) 14 and 39

(b) 234 and 165

(c) 1739 and 9923

(d) 471 and 562

(e) 23,771 and 19,945

(f) -4357 and 3754

Solution 10.

(a)

$$\begin{aligned}
39 &= 14 \times 2 + 11\\
14 &= 11 \times 1 + 3\\
11 &= 3 \times 3 + 2\\
3 &= 2 \times 1 + 1.
\end{aligned}$$

Therefore the gcd is 1. Now,

$$\begin{aligned}
1 &= 3 - 2 \times 1\\
&= 3 - (11 - 3 \times 3) \times 1\\
&= 3 \times 4 - 11 \times 1\\
&= (14 - 11) \times 4 - 11 \times 1\\
&= 14 \times 4 - 11 \times 5\\
&= 14 \times 4 - (39 - 14 \times 2) \times 5\\
&= 14 \times 14 - 39 \times 5.
\end{aligned}$$

Therefore if $r = 14$ and $s = -5$ we have $14a + 39b = 1$.

(b)

(c)

$$\begin{aligned}
9923 &= 1739 \times 5 + 1228\\
1739 &= 1228 \times 1 + 511\\
1228 &= 511 \times 2 + 206\\
511 &= 206 \times 2 + 99\\
206 &= 99 \times 2 + 8\\
99 &= 8 \times 12 + 3\\
8 &= 3 \times 2 + 2\\
3 &= 2 \times 1 + 1.
\end{aligned}$$

Therefore the gcd is 1.

Now,

$$\begin{aligned}
1 &= 3 - 2 \times 1 \\
&= 3 - (8 - 3 \times 2) \times 1 \\
&= 3 \times 3 - 8 \times 1 \\
&= (99 - 8 \times 12) \times 3 - 8 \times 1 \\
&= 99 \times 3 - 8 \times 37 \\
&= 99 \times 3 - (206 - 99 \times 2) \times 37 \\
&= 99 \times 77 - 206 \times 37 \\
&= (511 - 206 \times 2) \times 77 - 206 \times 37 \\
&= 511 \times 77 - 206 \times 191 \\
&= 511 \times 77 - (1228 - 511 \times 2) \times 191 \\
&= 511 \times 459 - 1228 \times 191 \\
&= (1739 - 1228 \times 1) \times 459 - 1228 \times 191 \\
&= 1739 \times 459 - 1228 \times 650 \\
&= 1739 \times 459 - (9923 - 1739 \times 5) \times 650 \\
&= 1739 \times 3709 - 9923 \times 650.
\end{aligned}$$

Therefore if $r = 3709$ and $s = -650$ we have $1739r + 9923s = 1$.

(d)

(e)

$$\begin{aligned}
23771 &= 19945 \times 1 + 3826 \\
19945 &= 3826 \times 5 + 815 \\
3826 &= 816 \times 4 + 566 \\
815 &= 566 \times 1 + 249 \\
566 &= 249 \times 2 + 68 \\
249 &= 68 \times 3 + 45 \\
68 &= 45 \times 1 + 23 \\
45 &= 23 \times 1 + 22 \\
23 &= 22 \times 1 + 1
\end{aligned}$$

Therefore the gcd is 1.

Now,

$$\begin{aligned}
1 &= 23 - 22 \times 1\\
&= 23 - (45 - 23 \times 1) \times 1\\
&= 23 \times 2 - 45 \times 1\\
&= (68 - 45 \times 1) \times 2 - 45 \times 1\\
&= 68 \times 2 - 45 \times 3\\
&= 68 \times 2 - (249 - 68 \times 3) \times 3\\
&= 68 \times 11 - 249 \times 3\\
&= (566 - 249 \times 2) \times 11 - 249 \times 3\\
&= 566 \times 11 - 249 \times 25\\
&= 566 \times 11 - (815 - 566 \times 1) \times 25\\
&= 566 \times 36 - 815 \times 25\\
&= (3826 - 815 \times 4) \times 36 - 815 \times 25\\
&= 3826 \times 36 - 815 \times 169\\
&= 3826 \times 36 - (19945 - 3826 \times 5) \times 169\\
&= 3826 \times 881 - 19945 \times 169.\\
&= (23771 - 19945 \times 1) \times 881 - 19945 \times 169\\
&= 23771 \times 881 - 19945 \times 1050.
\end{aligned}$$

Therefore if $r = 881$ and $s = -1050$ we have $23771r + 19945s = 1$.

(f)

$$\begin{aligned}
-4357 &= 3754 \times (-2) + 3151\\
3754 &= 3151 \times 1 + 603\\
3151 &= 603 \times 5 + 136\\
603 &= 136 \times 4 + 59\\
136 &= 59 \times 2 + 18\\
59 &= 18 \times 3 + 5\\
18 &= 5 \times 3 + 3\\
5 &= 3 \times 1 + 2\\
3 &= 2 \times 1 + 1.
\end{aligned}$$

Therefore the gcd is 1.

Now,

$$\begin{aligned}
1 &= 3 - 2 \times 1\\
&= 3 - (5 - 3 \times 1) \times 1\\
&= 3 \times 2 - 5 \times 1\\
&= (18 - 5 \times 3) \times 2 - 5 \times 1\\
&= 18 \times 2 - 5 \times 7\\
&= 18 \times 2 - (59 - 18 \times 3) \times 7\\
&= 18 \times 23 - 59 \times 7\\
&= (136 - 59 \times 2) \times 23 - 59 \times 7\\
&= 136 \times 23 - 59 \times 53\\
&= 136 \times 23 - (603 - 136 \times 4) \times 53\\
&= 136 \times 235 - 603 \times 53\\
&= (3151 - 603 \times 5) \times 235 - 603 \times 53\\
&= 3151 \times 235 - 603 \times 1228\\
&= 3151 \times 235 - (3754 - 3151 \times 1) \times 1228\\
&= 3151 \times 1463 - 3754 \times 1228\\
&= (-4357 - 3754 \times (-2)) \times 1463 - 3754 \times 1228\\
&= (-4357) \times 1463 + 3754 \times 1698
\end{aligned}$$

Therefore if $r = 1463$ and $s = 1698$ we have $(-4357)r + 3754s = 1$.

Problem 11. (**Exercise 23**)
Define the ***least common multiple*** of two nonzero integers a and b, denoted by $lcm(a, b)$, to be the nonnegative integer m such that both a and b divide m, and if a and b divide any other integer n, then m also divides n. Prove that any two integers a and b have a unique least common multiple.

Solution 11.

Problem 12. (**Exercise 27**)
Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Solution 12. Since $a \mid bc$, there exists an integer k such that $ak = bc$. Since $\gcd(a, b) = 1$, there exist integers r and s such that $ar + bs = 1$. Now multiply by c and we get

$$\begin{aligned}
arc + bcs &= c\\
arc + (ak)s &= c\\
a(rc + ks) &= c.
\end{aligned}$$

Since $rc + ks$ is an integer, then $a \mid c$.