Homework 4 Solutions

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1 Chapter 5

Problem 1. (**Exercise 1**)
Write the following permutations in cycle notation.

(a)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

Solution 1.

(a) (12453).

(b) To turn in.

(c) (13)(25).

(d) To turn in.

Problem 2. (**Exercise 2**)
Compute each of the following.

(a) (1345)(234)

(b) (12)(1253)

(c) (143)(23)(24)

(d) (1423)(34)(56)(1324)

(e) (1254)(13)(25)

(g) $(1254)^{-1}(123)(45)(1254)$

(n) $(12537)^{-1}$

Solution 2.

(a) $(1345)(234) = (135)(24)$.

(b) To turn in.

(c) $(143)(23)(24) = (14)(23)$.

(d) To turn in.

(e) $(1254)(13)(25) = (1324)$.

(g) $(1254)^{-1}(123)(45)(1254) = (4521)(123)(45)(1254) = (134)(25)$.

(n) $(12537)^{-1} = (73521) = (35217) = (21735) = (17352)$.

Problem 3. (**Exercise 3**)
Express the following permutations as products of transpositions and identify them as even or odd.

(a) (14356)

(b) $(156)(234)$

(c) $(1426)(142)$

(d) $(17254)(1423)(154632)$

Solution 3.

(a) $(14356) = (16)(15)(13)(14)$. The permutation is even.

(b) To turn in.

(c) $(1426)(142) = (16)(12)(14)(12)(14)$. The permutation is odd.

(d) To turn in.

Problem 4. (**Exercise 7**)
Find all possible orders of elements in S_7 and A_7.

Solution 4. To turn in.

Problem 5. (**Exercise 8**)
Show that A_{10} contains an element of order 15.

Solution 5. Let $\sigma = (12345)(678)$. Since $\sigma = (15)(14)(13)(12)(68)(67)$, σ is an even permutation, so $\sigma \in A_{10}$. Now, let's show that σ has order 15. Since (12345) and (678) are disjoint, then they commute, so $\sigma^n = (12345)^n(678)^n$ for all integers n. Since (12345) is a cycle with 5 elements $(12345)^n = id$ if and only if $5 \mid n$. Similarly, $(678)^n = id$ if and only if $3 \mid n$. Therefore $\sigma^n = id$ if and only if $15 \mid n$. Therefore the order of σ is 15.

Problem 6. (**Exercise 13**)
Let $\sigma = \sigma_1 \cdots \sigma_m \in S_n$ be the product of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \ldots, \sigma_m$.

Solution 6. To turn in.

Problem 7. (**Exercise 17**)
Prove that S_n is nonabelian for $n \geq 3$.

Solution 7. Since $n \geq 3$, then $\sigma = (12)$ and $\tau = (13)$ are both in S_n. Now, $\sigma \circ \tau = (12)(13) = (132)$ while $\tau \circ \sigma = (13)(12) = (123)$. Since $(132) \neq (123)$, then $\sigma\tau \neq \tau\sigma$, so S_n is nonabelian.

Problem 8. (**Exercise 27**)
Let G be a group and define a map $\lambda_g : G \to G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G.

Solution 8. A function f is a permutation of G, if $f : G \to G$ and f is a bijection. λ_g is a function from G to G, so to prove that it is a permutation, we must prove λ_g is a bijection.

Let's start by proving λ_g is one-to-one. Suppose $g_1, g_2 \in G$ and $\lambda_g(g_1) = \lambda_g(g_2)$. Then $gg_1 = gg_2$. By left-cancellation, we can conclude that $g_1 = g_2$. Therefore λ_g is one-to-one.

Now let's prove that λ_g is onto. Suppose $h \in G$. Then $g^{-1}h \in G$ since G is a group and $\lambda_g(g^{-1}h) = g(g^{-1}h) = h$. So λ_g is onto.

Since λ_g is a bijection, λ_g is a permutation of G.

Problem 9. (**Exercise 30**)
Let $\tau = (a_1, a_2, \ldots, a_k)$ be a cycle of length k.

(a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$$

is a cycle of length k.

(b) Let μ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

Solution 9.

(a) By multiplying by σ on the right, we can see that (a) is true if and only if

$$\sigma\tau = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma.$$

So let's prove this:

If $x \notin \{a_1, a_2, \ldots, a_k\}$, then $\sigma\tau(x) = \sigma(x)$ because $\tau(x) = x$. On the other hand $(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma(x) = \sigma(x)$, because the cycle $(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$ only acts on elements of the form $\sigma(a_i)$ and fixes everything else. Since $x \notin \{a_1, \ldots, a_k\}$, then the cycle fixes $\sigma(x)$. So when $x \notin \{a_1, a_2, \ldots, a_k\}$,

$$\sigma\tau(x) = \sigma(x) = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma(x).$$

If $x \in \{a_1, a_2, \ldots, a_k\}$, then $x = a_i$ for some $i \in \{1, 2, \ldots, k\}$. If $i \neq k$, then $\tau(a_i) = a_{i+1}$ so $\sigma\tau(x) = \sigma\tau(a_i) = \sigma(a_{i+1})$ and $(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma(a_i) = \sigma(a_{i+1})$. If $i = k$, then $\tau(a_k) = a_1$, so $\sigma\tau(a_k) = \sigma(a_1)$ and $(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma(a_k) = \sigma(a_1)$.

Therefore $\sigma\tau(x) = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))\sigma(x)$ for all x, hence the two functions are the same.

(b) Suppose $\mu = (b_1, b_2, \ldots, b_k)$. Now let σ be the permutation that satisfies $\sigma(a_i) = b_i$ and $\sigma(x) = x$ otherwise. Then by (a),

$$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)) = (b_1, b_2, \ldots, b_k) = \mu.$$

Problem 10. (**Exercise 33**)
Let $\alpha \in S_n$ for $n \geq 3$. If $\alpha\beta = \beta\alpha$ for all $\beta \in S_n$, prove that α must be the identity permutation; hence, the center of S_n is the trivial subgroup.

Solution 10. To turn in.