Homework 6 Solutions

Enrique Treviño

October 17, 2014

1 Chapter 9

Problem 1. (**Exercise 1**)
Prove that $\mathbb{Z} \cong n\mathbb{Z}$ for $n \neq 0$.

Solution 1. To turn in.

Problem 2. (**Exercise 3**)
Prove or disprove: $\mathbb{Z}_8^\times \cong \mathbb{Z}_4$.

Solution 2. $\mathbb{Z}_4$ is cyclic, yet $\mathbb{Z}_8^\times$ is not cyclic. Therefore they are not isomorphic.

Problem 3. (**Exercise 5**)
Show that $\mathbb{Z}_5^\times$ is isomorphic to $\mathbb{Z}_{10}^\times$, but $\mathbb{Z}_{12}^\times$ is not.

Solution 3. $\langle 2 \rangle = \{2, 4, 3, 1\} = \mathbb{Z}_5^\times$, so $\mathbb{Z}_5^\times$ is cyclic of order 4. $\mathbb{Z}_{10}^\times = \{1, 3, 7, 9\} = \{3, 9, 7, 1\} = \langle 3 \rangle$ is also cyclic of order 4. Therefore they are both isomorphic to $\mathbb{Z}_4$, so they are isomorphic to each other. $\mathbb{Z}_{12}^\times$ is not cyclic since all of its non-identity elements have order 2

Alternative Solution: Let $\phi : \mathbb{Z}_5^\times \to \mathbb{Z}_{10}^\times$ be defined by $\phi(1) = 1$, $\phi(2) = 3$, $\phi(4) = 9$, $\phi(3) = 7$. Then ϕ is a bijection. We can then verify that $\phi(ab \bmod 5) = \phi(a)\phi(b) \bmod 10$ by checking all 16 possible pairs $a, b \in \{1, 2, 3, 4\}$.

An alternative proof that $\mathbb{Z}_{12}^\times$ is not isomorphic to $\mathbb{Z}_5^\times$ is the following: Suppose that $\phi : \mathbb{Z}_5^\times \to \mathbb{Z}_{12}^\times$ is an isomorphism. Let $\phi(2) = a$. Then $\phi(4) = a^2$, $\phi(3) = a^3$ and $\phi(1) = a^4 = 1$. Then a, a^2, a^3 and 1 are all different. Yet $a^2 = 1$ for any $a \in \mathbb{Z}_{12}^\times$. Therefore no isomorphism exists.

Problem 4. (**Exercise 8**)
Prove that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$.

Solution 4. To turn in.

Problem 5. (**Exercise 11**)
Find five non-isomorphic groups of order 8 (prove that they are non-isomorphic).

Solution 5. Four easy groups to find are $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_4$. D_4 is non-abelian so it clearly is different from the rest. $\mathbb{Z}_8$ is the only cyclic group so that makes it different from the rest. $\mathbb{Z}_4 \times \mathbb{Z}_2$ has an element of order 4 since $|(1, 0)| = 4$, yet every nontrivial element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2. Therefore they are different.

The last group of order 8 is a special group that took Hamilton many years to find. The group is called the quaternion group and it has the following representation:

$$Q = \{1, i, j, k, -i, -j, -k, -1\},$$

with the operations $i^2 = j^2 = k^2 = ijk = -1$ and $(-1)^2 = 1$. 1 is the identity and -1 commutes with all elements.

With these operations we can deduce the rest of the operations. For example $ijk = -1$ and $k^2 = -1$ therefore $ijk^2 = (-1)(k)$, so $-ij = -k$, so $ij = k$. Also, $(ij)(ji) = i(j^2)i = i(-i) = -i^2 = 1$. So ji is the inverse of ij. Since $ij = k$, then $ji = -k$. We can also find ik. Indeed $k = ij$, so $ik = i(ij) = -j$. With similar reasoning we can find all of them and form the Cayley table:

$\times$	1	i	j	k	$-i$	$-j$	$-k$	-1
1	1	i	j	k	-1	$-j$	$-k$	-1
i	i	-1	k	$-j$	1	$-k$	j	$-i$
j	j	$-k$	-1	i	k	1	$-i$	$-j$
k	k	j	$-i$	-1	$-j$	i	1	$-k$
$-i$	$-i$	1	$-k$	j	-1	k	$-j$	i
$-j$	$-j$	k	1	$-i$	$-k$	-1	i	j
$-k$	$-k$	$-j$	i	1	j	$-i$	-1	k
-1	-1	$-i$	$-j$	$-k$	i	j	k	1

Note that $ij = k$ and $ji = -k$, therefore Q is non-abelian. Therefore if it's isomorphic to any of the others it can only be isomorphic to D_4. It is not isomorphic to D_4 because D_4 only has 2 elements of order 4 (r and r^3) where as Q has 6 elements of order $4(i, j, k, -i, -j, -k)$. Indeed if we suppose an isomorphism $\phi : D_4 \to Q$ exists, then if $a \in D_4$ has order n, then $\phi(a) \in Q$ has order n (that is because $\phi(a^k) = \phi(a)^k$, so $a^k = id \Leftrightarrow n|k$ and $\phi(a^k) = id \Leftrightarrow a^k = id \Leftrightarrow n|k$. Since $\phi(a^k) = \phi(a)^k$, then a and $\phi(a)$ have the same orders in their respective groups. In particular that implies that D_4 has the same number of elements of order 4 as Q. But this is not true. Therefore $D_4 \ncong Q$. Therefore we have 5 non-isomorphic groups of order 8.

Problem 6. (**Exercise 16**)
Find the order of each of the following elements.

(a) $(3, 4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$

(b) $(6, 15, 4)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{45} \times \mathbb{Z}_{24}$

(c) $(5, 10, 15)$ in $\mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_{25}$

(d) $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$

Solution 6.

(a) The order of 3 in $\mathbb{Z}_4$ is 4 and the order of 4 in $\mathbb{Z}_6$ is 3. Therefore the order of $(3, 4) = lcm(4, 3) = 12$.

(b) To turn in.

(c) The orders of 5, 10 and 15 in $\mathbb{Z}_{25}$ are 5, 5 and 5 respectively. Therefore the order of (5, 10, 15) is $lcm(5, 5, 5) = 5$.

(d) To turn in.

Problem 7. (**Exercise 17**)
Prove that D_4 cannot be the internal direct product of two of its proper subgroups.

Solution 7. To turn in.

Problem 8. (**Exercise 22**)
Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5 respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

Solution 8. To be able to prove this statement we need to prove that $G = HK$ and that $H \cap K = \{e\}$. Let's start by proving that $H \cap K = \{e\}$. Suppose $x \in H \cap K$. Then $x \in H$ so $x^4 = e$ and $x \in K$ so $x^5 = e$. Since $x^4 = e$, then $x^5 = x^4x = x$. Since $x^5 = e$ and $x^5 = x$, then $x = e$. Therefore $H \cap K = \{e\}$.

Now let's show that $HK = G$. H has 4 elements and K has 5 elements so there are 20 possible elements of the form hk with $h \in H$ and $k \in K$. Since H and K are subsets of G, all of these hk are elements of G. Since G has 20 elements, if all of the hk's are different, then $HK = G$. The only way we fail to show $HK = G$ is if we have $h_1 k_1 = h_2 k_2$ with either $h_1 \neq h_2$ or $k_1 \neq k_2$. So suppose $h_1 k_1 = h_2 k_2$. Then $h_2^{-1} h_1 k_1 = k_2$, so

$$h_2^{-1} h_1 = k_2 k_1^{-1}.$$

Therefore $h_2^{-1} h_1 \in H \cap K$ and $k_2 k_1^{-1} \in H \cap K$. Since $H \cap K = \{e\}$, then $h_2^{-1} h_1 = e$ and $k_2 k_1^{-1} = e$. Therefore $h_1 = h_2$ and $k_1 = k_2$. That means that all of the hk's are different so $G = HK$. Therefore $G \cong H \times K$.

Problem 9. (**Exercise 23**)
Prove or disprove the following assertion. Let G, H, and K be groups. If $G \times K \cong H \times K$, then $G \cong H$.

Solution 9. To turn in.

Problem 10. (**Exercise 33**)
Write out the permutations associated with each element of S_3 in the proof of Cayley's Theorem.

Solution 10. To turn in.