Homework 7 Solutions

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1 Chapter 10

Problem 1. (**Exercise 1**)
For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

(a) $G = S_4$ and $H = A_4$

(b) $G = A_5$ and $H = \{(1), (123), (132)\}$

(c) $G = S_4$ and $H = D_4$

(d) $G = Q_8$ and $H = \{1, -1, I, -I\}$

(e) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution 1.

(a) Since $[S_4 : A_4] = 2$, then the two cosets are A_4 and B_4 (set of odd permutations). Then if $\sigma \in A_4$, then $\sigma A_4 = A_4 = A_4\sigma$. If $\sigma \notin A_4$, then $\sigma A_4 \neq A_4$, so $\sigma A_4 = B_4$. Similarly $A_4\sigma \neq A_4$, so $A_4\sigma = B_4$. Therefore $\sigma A_4 = A_4\sigma$. Therefore A_4 is a normal subgroup of S_4. The Cayley table is pretty simple:

$\circ$	A_4	B_4
A_4	A_4	B_4
B_4	B_4	A_4

(b) To turn in.

(c) To turn in.

(d) As a reminder

$$Q_8 = \{1, -1, i, j, k, -i, -j, -k\},$$

with $i^2 = j^2 = k^2 = ijk = -1$. We will show $H \trianglelefteq Q_8$ by showing that if $g \in Q_8$ and $h \in H$, then $ghg^{-1} \in H$. First note that if $g \in H$, then since H is a subgroup, then $ghg^{-1} \in H$. So we can assume $g \notin H$. We also know $g1g^{-1} = 1$ and $g(-1)g^{-1} = -1$. It's also easy to see that $g(-i)g^{-1} = -gig^{-1}$. So we need only show that if $g \notin H$ and $h = i$, then $ghg^{-1} \in H$. We have four cases: (we will use that $ij = k$ since $ijk = -1$ and $k^2 = -1$, $jk = i$ since $ijk = -1$ and $i^2 = -1$. Also the inverse of g is $-g$ for any $1 \neq g \in Q_8$)

1. $g = j$. Then $gig^{-1} = jij^{-1} = ji(-j) = -(jij) = -jk = -i \in H$.
2. $g = -j$. Then $gig^{-1} = (-j)i(-j)^{-1} = (-j)i(j) = -(jij) = -jk = -i \in H$.
3. $g = k$. Then$gig^{-1} = kik^{-1} = ki(-k) = -(kik) = -kj = i \in H$.
4. $g = -k$. Then$gig^{-1} = (-k)i(-k)^{-1} = (-k)i(k) = -(kik) = -kj = i \in H$.

Therefore $H \trianglelefteq Q_8$.

One could also prove it as in part a, by using that the index of H in Q_8 is 2.

Now the Cayley table of G/H:

$\circ$	H	$\{j,-j,k,-k\}$
H	H	$\{j,-j,k,-k\}$
$\{j,-j,k,-k\}$	$\{j,-j,k,-k\}$	H

(e) To turn in.

Problem 2. (**Exercise 2**)
Find all the subgroups of D_4. Which subgroups are normal? What are all the factor groups of D_4 up to isomorphism?

Solution 2. To turn in.

Problem 3. (**Exercise 5**)
Show that the intersection of two normal subgroups is a normal subgroup.

Solution 3. Let N_1 and N_2 be normal subgroups of G. Let $N = N_1 \cap N_2$. Let's show that $N \trianglelefteq G$. We will prove it by showing that $gng^{-1} \in N$ for any $g \in G$ and $n \in N$. Indeed, let $g \in G$ and $n \in N$. Since $n \in N_1 \cap N_2$, then $n \in N_1$. Since N_1 is normal, then $gng^{-1} \in N_1$. Therefore $gng^{-1} \in N_1$. But also $n \in N_2$ and N_2 is normal, so $gng^{-1} \in N_2$. Therefore $gng^{-1} \in N_1$ and $gng^{-1} \in N_2$, so $gng^{-1} \in N_1 \cap N_2 = N$. So $N \trianglelefteq G$.

Problem 4. (**Exercise 9**)
Prove or disprove: If H and G/H are cyclic, then G is cyclic.

Solution 4. To turn in.

Problem 5. (**Exercise 10**)
Let H be a subgroup of index 2 of a group G. Prove that H must be a normal subgroup of G. Conclude that S_n is not simple for $n \geq 3$.

Solution 5. To turn in.

Problem 6. (**Exercise 11**)
If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Solution 6. To turn in.

Problem 7. (**Exercise 13**)
Recall that the **center** of a group G is the set

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G \}.$$

(a) Calculate the center of S_3.

(b) Calculate the center of $GL_2(\mathbb{R})$.

(c) Show that the center of any group G is a normal subgroup of G.

(d) If $G/Z(G)$ is cyclic, show that G is abelian.

Solution 7.

(a) To turn in.

(b) We want to find all invertible matrices M that satisfy $AM = MA$ for any invertible matrix A. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now since M commutes with all elements of $GL_2(\mathbb{R})$, then in particular, it commutes with

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore

$$JM = \begin{pmatrix} c & d \\ a & b \end{pmatrix} = MJ = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

Therefore $a = d$ and $b = d$. Therefore M is of the form

$$M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Now, M should also commute with

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$KM = \begin{pmatrix} (a+b) & (a+b) \\ a & b \end{pmatrix} = MK = \begin{pmatrix} (a+b) & a \\ (a+b) & b \end{pmatrix}.$$

Therefore $a + b = a$, so $b = 0$. Therefore

$$M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

And indeed if

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(\mathbb{R}),$$

then

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & aw \end{pmatrix},$$

and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & aw \end{pmatrix}.$$

So M commutes with every element of $GL_2(\mathbb{R})$, so M is in the center. Yet every element in the center must be of this form. Therefore

$$Z(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}$$

(c) To turn in.

(d) Suppose that $G/Z(G)$ is cyclic. Then $G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$. Let $g_1, g_2 \in G$. We want to show that $g_1 g_2 = g_2 g_1$ (to show G is abelian). Since $G/Z(G) = \langle gZ(G) \rangle$, then $g_1 Z(G) = g^{k_1} Z(G)$ for some integer k_1 and $g_2 Z(G) = g_2^{k_2} Z(G)$ for some integer k_2. Then $g_1 \in g^{k_1} Z(G)$, so there exists an $x \in Z(G)$ such that $g_1 = g^{k_1} x$. Similarly, there exists a $y \in Z(G)$ such that $g_2 = g^{k_2} y$. Therefore

$$\begin{aligned} g_1 g_2 &= (g^{k_1} x)(g^{k_2} y) \\ &= (x g^{k_1})(g^{k_2} y) \\ &= x(g^{k_1+k_2})y \\ &= xyg^{k_1+k_2}. \end{aligned}$$

We used that x and y commute with any element of G. Similarly:

$$\begin{aligned} g_2 g_1 &= (g^{k_2} y)(g^{k_1} x) \\ &= (y g^{k_2})(g^{k_1} x) \\ &= y(g^{k_2+k_1})x \\ &= yxg^{k_1+k_2} \\ &= xyg^{k_1+k_2}. \end{aligned}$$

Therefore $g_1 g_2 = g_2 g_1$.