Homework 8 Solutions

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1 Chapter 11

Problem 1. (**Exercise 4**)
Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be given by $\phi(n) = 7n$. Prove that ϕ is a group homomorphism. Find the kernel and the image of ϕ.

Solution 1. To turn in.

Problem 2. (**Exercise 5**)
Describe all of the homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

Solution 2. Let ϕ be a homomorphism from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$. Suppose $\phi(1) = k$ for some $k \in \mathbb{Z}_{18}$. Then since ϕ is a homomorphism $\phi(m) = m\phi(1) = mk$. So $\phi(1)$ determines $\phi(m)$ for all $m \in \mathbb{Z}_{24}$. Now since $24 \equiv 0 \bmod 24$, then $24k = \phi(24) = \phi(0) = 0$. But that means $24k \equiv 0 \bmod 18$. Therefore $8k \equiv 0 \bmod 6$. Therefore $4k \equiv 0 \bmod 3$. Therefore $k \equiv 0 \bmod 3$. So the only options are $k = 0, 3, 6, 9, 12, 15$. Each of one of these gives a homomorphism.

So the homomorphisms are $\phi(m) = mk \bmod 18$ for each $k = 0, 3, 6, 9, 12, 15$. Let's write out the case $k = 3$ to show that homomorphism. The others are built similarly:

$$\begin{array}{rrrrrr}
\phi(0) = 0, & \phi(1) = 3, & \phi(2) = 6, & \phi(3) = 9, & \phi(4) = 12, & \phi(5) = 15, \\
\phi(6) = 0, & \phi(7) = 3, & \phi(8) = 6, & \phi(9) = 9, & \phi(10) = 12, & \phi(11) = 15, \\
\phi(12) = 0, & \phi(13) = 3, & \phi(14) = 6, & \phi(15) = 9, & \phi(16) = 12, & \phi(17) = 15, \\
\phi(18) = 0, & \phi(19) = 3, & \phi(20) = 6, & \phi(21) = 9, & \phi(22) = 12, & \phi(23) = 15.
\end{array}$$

Problem 3. (**Exercise 6**)
Describe all of the homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{12}$.

Solution 3. To turn in.

Problem 4. (**Exercise 7**)
In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

(a) List the elements in HN (we usually write $H + N$ for these additive groups) and $H \cap N$.

(b) List the cosets in HN/N, showing the elements in each coset.

(c) List the cosets in $H/(H \cap N)$, showing the elements in each coset.

(d) Give the correspondence between HN/N and $H/(H \cap N)$ described in the proof of the Second Isomorphism Theorem.

Solution 4. To turn in.

Problem 5. (**Exercise 9**)
If $\phi : G \to H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is also abelian.

Solution 5. Suppose that $\phi(g_1), \phi(g_2) \in \phi(G)$. Since $g_1, g_2 \in G$ and G is abelian, then $g_1 g_2 = g_2 g_1$. Therefore

$$\phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1).$$

Therefore $\phi(G)$ is abelian.

Problem 6. (**Exercise 10**)
If $\phi : G \to H$ is a group homomorphism and G is cyclic, prove that $\phi(G)$ is also cyclic.

Solution 6. To turn in.

Problem 7. (**Exercise 11**)
Show that a homomorphism defined on a cyclic group is completely determined by its action on the generator of the group.

Solution 7. Suppose $\phi : G \to H$ is a group homomorphism and that $G = \langle g \rangle$ is cyclic. Let $\phi(g) = h$ for some $h \in H$. Since ϕ is a group homomorphism then $\phi(g^k) = \phi(g)^k = h^k$. Now if $g' \in G$, then $g' = g^k$ for some integer k. Then

$$\phi(g') = \phi(g^k) = \phi(g)^k = h^k.$$

Therefore the homomorphism is determined by the action on the generator since every other value can be written in terms of the value on the generator.

Problem 8. (**Exercise 12**)
Let G be a group of order p^2, where p is a prime number. If H is a subgroup of G of order p, show that H is normal in G. Prove that G must be abelian.

Solution 8. To turn in.

Problem 9. (**Exercise 17**)
If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

Solution 9. To turn in.