Integrating with series expansion of e^x

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Let's integrate $\int xe^{2x}\,\mathrm{d}x$.

The standard way is by integration by parts, using that $u = x$ and $\mathrm{d}v = e^{2x}\mathrm{d}x$. Then we get $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{e^{2x}}{2}$. Therefore

$$\int xe^{2x}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Let's do it a different way using the series expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Then

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

Therefore

$$\begin{aligned}\int xe^{2x}\mathrm{d}x &= \int x + 2x^2 + \frac{2^2x^3}{2!} + \frac{2^3x^4}{3!} + \cdots \\ &= \frac{x^2}{2} + \frac{2x^3}{3} + \frac{2^2x^4}{4\cdot 2!} + \frac{2^3x^5}{5\cdot 3!} + \cdots\end{aligned}$$

Let's look at the n-th term in the series:

$$\frac{2^n x^{n+2}}{(n+2)\cdot n!}.$$

By multiplying and dividing by 4, and multiplying and dividing by $(n+1)$, we get

$$\frac{1}{4}\left(\frac{(2x)^{n+2}(n+1)}{(n+2)!}\right).$$

But then, $n+1 = (n+2) - 1$, so

$$\frac{n+1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}.$$

Therefore, the n-th term in the series is

$$\begin{aligned}\frac{2^n x^{n+2}}{(n+2)\cdot n!} &= \frac{1}{4}\frac{(2x)^{n+2}(n+1)}{(n+2)!} \\ &= \frac{1}{4}\frac{(2x)^{n+2}}{(n+1)!} - \frac{1}{4}\frac{(2x)^{n+2}}{(n+2)!} \\ &= \frac{x}{2}\frac{(2x)^{n+1}}{(n+1)!} - \frac{1}{4}\frac{(2x)^{n+2}}{(n+2)!}.\end{aligned}$$

So the integral of xe^{2x} can be re-written as

$$\frac{x}{2}\left(\frac{(2x)^1}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right) - \frac{1}{4}\left(\frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots\right).$$

Note that both expressions in the parenthesis are almost the same as

$$1 + 2x + \frac{(2x)^2}{2!} + \cdots = e^{2x}.$$

In fact, the first expression in the parenthesis is $e^{2x} - 1$ and the second one is $e^{2x} - 1 - 2x$, so we get

$$\int xe^{2x}\mathrm{d}x = \frac{x}{2}(e^{2x} - 1) - \frac{1}{4}(e^{2x} - 1 - 2x) + C = \frac{xe^{2x}}{2} - \frac{x}{2} - \frac{1}{4}e^{2x} + \frac{1}{4} + \frac{x}{2}.$$

Using that $1/4$ is a constant and that the $\frac{x}{2}$ terms cancel out, we conclude that

$$\int xe^{2x}\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$