Extra Problem Homework 6 (π-day special)

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Let $y(t)$ be the following function of t:

$$y(t) = \int_0^\infty e^{-x^2} \cos(2xt)\, dx.$$

(a) Prove that $y(t)$ satisfies the differential equation

$$y' + 2ty = 0.$$

(Hint: You have to compute y', then integrate it by parts).

(b) Find the general solution to this differential equation.

(c) In class we proved that

$$y(0) = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Use this initial condition to show that

$$y(t) = \frac{\sqrt{\pi}}{2} e^{-t^2}.$$

Solution:

$$y'(t) = -2\int_0^\infty x e^{-x^2} \sin(2xt)\, dx.$$

By parts, using that the integral of xe^{-x^2} is $(-1/2)e^{-x^2}$, we get that $y'(t)$ equals

$$\begin{aligned}
y'(t) &= \left(\sin(2xt)e^{-x^2}\right)\Big|_{x=0}^{x=\infty} - 2t\int_0^\infty e^{-x^2}\cos(2xt)\, dx \\
&= \left(\lim_{x\to\infty} \sin(2xt)e^{-x^2} - \sin(0)e^{-0^2}\right) - 2t\int_0^\infty e^{-x^2}\cos(2xt)\, dx \\
&= -2t\int_0^\infty e^{-x^2}\cos(2xt)\, dx \\
&= -2ty(t).
\end{aligned}$$

Therefore $y'(t) + 2ty(t) = 0$.

Now we can solve the differential equation using integrating factors or by separating. I will show the solution using separable variables:

$$\begin{aligned} \frac{dy}{dt} &= -2ty \\ \frac{dy}{y} &= -2t\,dt \\ \ln y &= -t^2 + C \\ y &= Ae^{-t^2}, \end{aligned}$$

for some constant A. Therefore

$$y(t) = Ae^{-t^2}.$$

Since $y(0) = A$ and $y(0) = \sqrt{\pi}/2$, we know that $A = \sqrt{\pi}/2$. Therefore

$$y(t) = \frac{\sqrt{\pi}}{2}e^{-t^2}.$$