$\boxed{10.1}$

(1) $y''+y=0, \quad y(0)=0, \quad y'(\pi)=1.$

$r^2+1=0$ so $r=i$, so $y=c_1\cos t + c_2 \sin t$.

$y(0)=0$ so $c_1=0$.

$y'(\pi)=1$ so $-c_2=1$, so $c_2=-1$.

Therefore $\boxed{y(t)=-\sin t}$

(5) $y''+y=x, \quad y(0)=0, \quad y(\pi)=0$

Homogeneous sol: $y=c_1\cos x + c_2 \sin x$.

Suppose $y=Ax+B$ Then $y''=0$ and $y=Ax+B$.

So $Ax+B=x$, so $A=1, B=0$.

So $y=x$ is a solution.

Then the general solution is $y(x)=x+c_1\cos x + c_2 \sin x$.

$y(0)=0$ implies $c_1=0$.

$y(\pi)=0$ implies $\pi-c_1=0$, so $c_1=\pi$.

So it has no solutions.

(10) $y''+3y=\cos x, \quad y'(0)=0, \quad y'(\pi)=0.$

Homogeneous: $r^2+3=0$ so $y(x)=c_1\cos(\sqrt{3}x)+c_2\sin(\sqrt{3}x)$

Let $y=A\cos x + B\sin x$.

Then $y'=-A\sin x + B\cos x$, so $y''=-A\cos x - B\sin x$

so $y''+3y= 2A\cos x + 2B\sin x = \cos x$. So $A=\frac{1}{2}, B=0$.

So $y(x)=c_1\cos(\sqrt{3}x)+c_2\sin(\sqrt{3}x)+\frac{1}{2}\cos x$

$y'(x) = -\sqrt{3}\,c_1 \sin(\sqrt{3}\,x) + \sqrt{3}\,c_2 \cos(\sqrt{3}\,x) - \frac{1}{2}\sin x.$

$y'(0) = \sqrt{3}\,c_2$, so $c_2 = 0$.

Then $y'(x) = -\sqrt{3}\,c_1 \sin(\sqrt{3}\,x) - \frac{1}{2}\sin(x)$,
so $y'(\pi) = -\sqrt{3}\,c_1 \sin(\sqrt{3}\,\pi)$, so $c_1 = 0$.

So $\boxed{y(x) = \frac{1}{2}\cos x}$

(14) $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$

$r^2 + \lambda = 0$ so $r = \sqrt{-\lambda}$ if $\lambda \leq 0$ and $r = \sqrt{\lambda}\, i$ if $\lambda > 0$.

Suppose $\lambda < 0$, then $y(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

so $y(0) = c_1 + c_2 = 0$
and $y'(\pi) = \sqrt{-\lambda}\,c_1 e^{\sqrt{-\lambda}\,\pi} - \sqrt{-\lambda}\,c_2 e^{-\sqrt{-\lambda}\,\pi} = 0$

Then $c_1 + c_2 = 0$ and $c_1 e^{\sqrt{-\lambda}\,\pi} = c_2 e^{-\sqrt{-\lambda}\,\pi}$

if $c_1 = 0 \Rightarrow c_2 = 0$ and λ is not an eigenvalue

Suppose $c_1 \neq 0$. Then $e^{\sqrt{-\lambda}\,\pi} = -e^{-\sqrt{-\lambda}\,\pi}$
but $e^{\sqrt{-\lambda}\,\pi} > 0$ and $e^{-\sqrt{-\lambda}\,\pi} > 0$.
Hence this is impossible.
So $\lambda < 0$ is not an eigenvalue.

If $\lambda = 0$, then $y'' = 0$, so $y(x) = Ax + B$. Then $y(0) = 0$, implies $B = 0$.
$y'(\pi) = 0$ implies $A = 0$. So $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$. Then $y(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$.

$y(0) = 0 \Rightarrow c_1 = 0$.

$y'(x) = -\sqrt{\lambda}\,c_1 \sin(\sqrt{\lambda}\,x) + \sqrt{\lambda}\,c_2 \cos(\sqrt{\lambda}\,x)$

$y'(\pi) = \sqrt{\lambda}\,c_2 \cos(\sqrt{\lambda}\,\pi) = 0$. So $\cos(\sqrt{\lambda}\,\pi) = 0$.

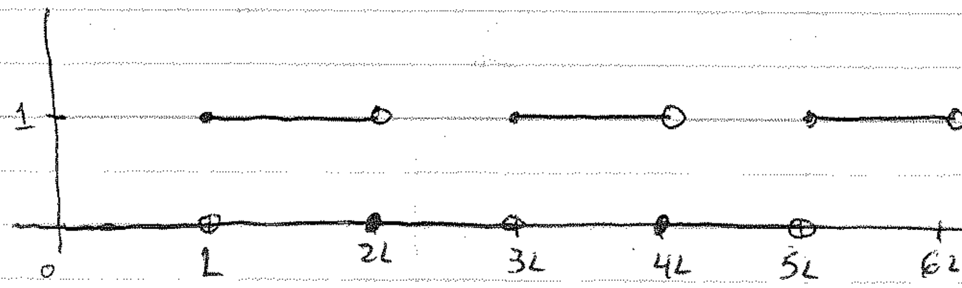
So $\sqrt{\lambda}\,\pi = \frac{\pi}{2} + k\pi = \left(\frac{2k+1}{2}\right)\pi$ for $k = 0, 1, 2, 3, \dots$

Then $\boxed{\lambda = \left(\frac{2k+1}{2}\right)^2}$ and $\boxed{y(x) = C \sin\left(\frac{2k+1}{2}\,x\right)}$

$\boxed{10.2}$

(14) $f(x) = \begin{cases} 1, & -L \le x < 0 \\ 0, & 0 \le x < L \end{cases}$; $f(x+2L) = f(x)$

a)



b) $\frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{0} 1\,dx = \frac{L}{L} = 1.$ So $a_0 = 1$

$$\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n x\pi}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\cos\left(\frac{n x\pi}{L}\right)dx = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0}$$

$$= 0 - \frac{L}{n\pi}\sin(n(-\pi)) = 0.$$

So $a_n = 0$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\sin\left(\frac{m\pi x}{L}\right)dx = \frac{-1}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\Big|_{-L}^{0}$$

$$= \frac{-1}{m\pi} + \frac{1}{m\pi}\cos(m\pi) = \frac{(-1)^m - 1}{m\pi}.$$

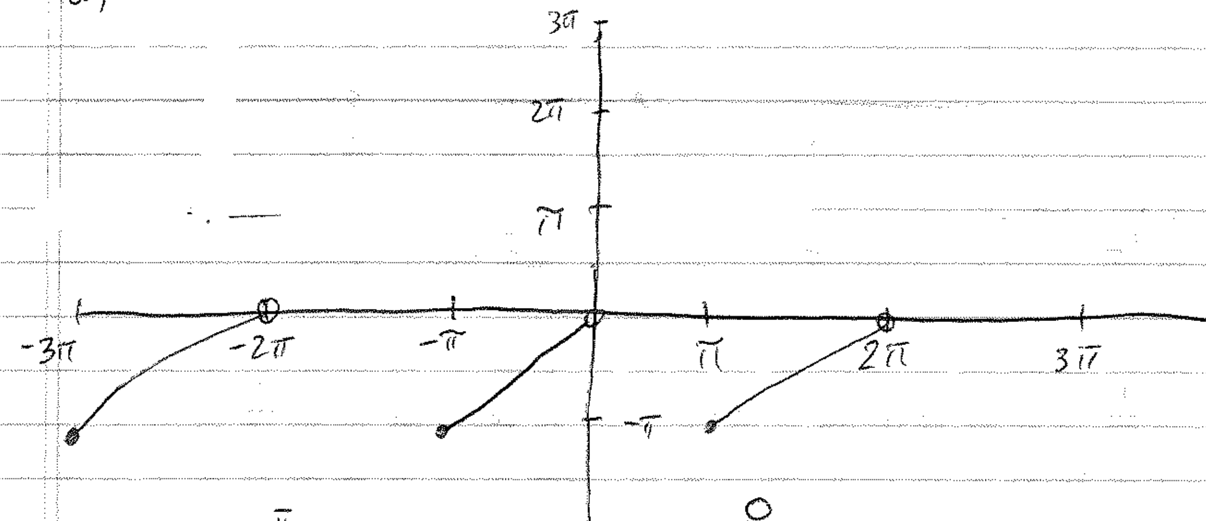
So $\boxed{f(x) = \frac{1}{2} + \sum_{m=1}^{\infty}\frac{(-1)^m - 1}{m\pi}\sin\left(\frac{m\pi x}{L}\right)} = \frac{1}{2} - \frac{2}{\pi}\sum_{\substack{m=1 \\ m \text{ is odd}}}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi x}{L}\right)$

or $f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\left(\frac{(2m-1)\pi x}{L}\right)$

(15) $f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}$; $f(x+2\pi) = f(x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\,\frac{x^2}{2}\Big|_{-\pi}^{0} = 0 - \frac{1}{\pi}\cdot\frac{\pi^2}{2} = -\frac{\pi}{2}$$

a)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx$$

$u = x$, $dv = \cos(nx)$

$du = dx$ $\quad v = \frac{1}{n}\sin(nx)$

$$a_n = \frac{1}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin(nx)\,dx\right)$$

$$= \frac{1}{\pi}\left(0 + \frac{1}{n^2}\cos(nx)\Big|_{-\pi}^{0}\right) = \frac{1}{\pi}\left(\frac{1}{n^2}\left(1-(-1)^n\right)\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2}{n^2\pi} & \text{if } n \text{ is even}\end{cases}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{m\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(mx)\,dx$$

$u = x$ $\quad dv = \sin(mx)$

$du = dx$ $\quad v = -\frac{1}{m}\cos(mx)$

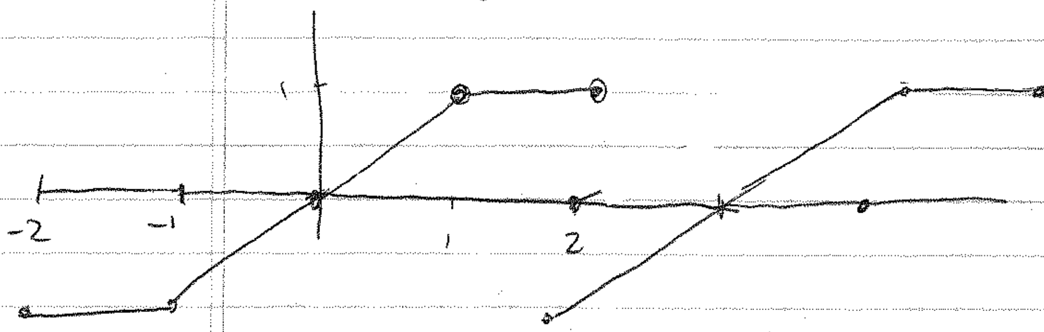
$$b_m = \frac{1}{\pi}\left(-\frac{1}{m}x\cos(mx)\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{1}{m}\cos(mx)\,dx\right)$$

$$= \frac{1}{m\pi}\left(0 - \pi\cos(m\pi)\right) + \frac{1}{m^2\pi}\sin(mx)\Big|_{-\pi}^{0} = \boxed{\frac{(-1)^{m+1}}{m}}$$

So $\boxed{f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(mx)}$

$\boxed{10.4}$

(16) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$, sine series, period 4.



$a_0 = 0 \;, \quad a_n = 0.$

$$b_m = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{m\pi x}{2}\right)dx$$

$$= \int_0^1 x\sin\left(\frac{m\pi x}{2}\right)dx + \int_1^2 \sin\left(\frac{m\pi x}{2}\right)dx$$

$u = x \qquad dv = \sin\left(\frac{m\pi x}{2}\right)$

$du = dx \qquad v = \frac{-2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)$

$$= \left.\frac{-2x}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\right|_0^1 + \int_0^1 \frac{2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)dx + \left.\left(\frac{-2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\right)\right|_1^2$$

$$= \frac{-2}{m\pi}\cos\left(\frac{m\pi}{2}\right) + \cancel{\frac{2(0)}{m\pi}}^{\,0} + \left.\left(\frac{2}{m\pi}\right)^2\sin\left(\frac{m\pi x}{2}\right)\right|_0^1 + \left(\frac{-2}{m\pi}\cos(m\pi) + \frac{2}{m\pi}\cos\left(\frac{m\pi}{2}\right)\right)$$

$$= \left(\frac{2}{m\pi}\right)^2\sin\left(\frac{m\pi}{2}\right) - \frac{2}{m\pi}\cos(m\pi)$$

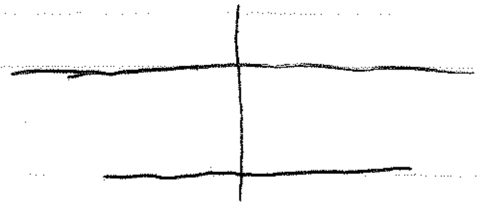
$$\boxed{f(x) = \sum_{m=1}^{\infty}\left(\left(\frac{2}{m\pi}\right)^2\sin\left(\frac{m\pi}{2}\right) - \frac{2}{m\pi}\cos(m\pi)\right)\sin\frac{m\pi x}{2}}$$

It can be simplified further using $\sin\left(\frac{m\pi}{2}\right) = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \equiv 1 \bmod 4 \\ -1 & \text{if } m \equiv 3 \bmod 4 \end{cases}$

and $\cos(m\pi) = (-1)^m$.

(17) $f(x)=1$, $0 \le x \le \pi$, cosine series period 2π

So even function



$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos\left(\frac{n\pi x}{2\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} \cos\left(\frac{nx}{2}\right)dx$$

If $n=0$, then $a_0 = \frac{1}{\pi}\int_0^{2\pi} 1\,dx = 2$.

If $n \neq 0$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{nx}{2}\right)\Big|_0^{2\pi} = \frac{2}{n\pi}\Big(\sin(2\pi n) - \sin(0)\Big) = 0.$$

Then $f(x) = \frac{a_0}{2} + 0 = 1$.