Homework 1 SOLUTIONS

SECTION 1.1

(7) $\frac{dy}{dt} = ay + b$ such that the solutions approach $y = 3$.

$\frac{dy}{dt} = 0$ for $y = 3$ so $3a + b = 0$. Therefore $a = -\frac{b}{3}$

We want $\frac{dy}{dt} < 0$ when $y > 3$, so $-\frac{b}{3} \cdot y + b < 0$ for $y > 3$

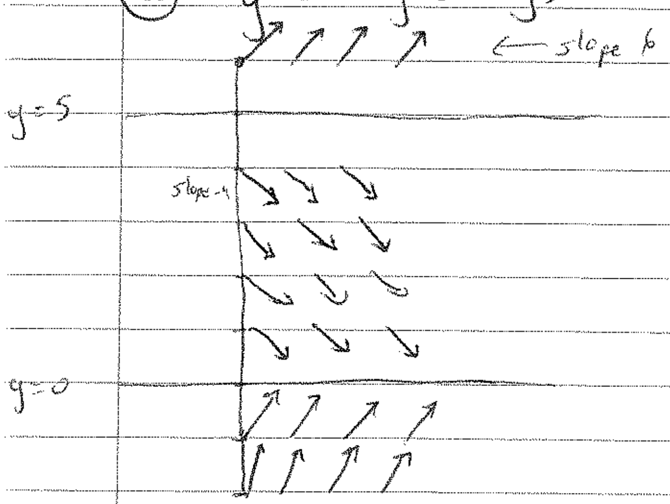
$-\frac{b}{3}(6) + b < 0$, so $-2b + b < 0$ so $b > 0$.

If $b > 0$, then $a < 0$.

After choosing any $b \geq 0$, let $a = -b/3$ and $\frac{dy}{dt} = ay + b$ satisfies the conditions.

To pick an example, let $b = 6$, so $\boxed{\frac{dy}{dt} = -2y + 6}$

(12) $y' = -y(5 - y)$



The equilibrium sols are $y = 5$ and $y = 0$.

Let y_0 be the initial value.

If $-\infty < y_0 < 5$, then y will go to 0 as $t \to \infty$.

If $y_0 > 5$, then $y \to \infty$ as $t \to \infty$.

If $y_0 = 5$, then $y = 5$ for all t.

SECTION 1.2

(8) $\frac{dp}{dt} = rp$

a) Rate r such that the population doubles in 30 days.

Let r have /day as its units.

Let p_0 be the initial population.

$\frac{dp}{p} = r\,dt$ so $\ln|p| = rt + C_0$ so $p(t) = Ce^{rt}$

for some constant C.

d_f $t = 0 \Rightarrow p(t) = Ce^0 = C$. So $C = p_0$.

Therefore $p(t) = p_0 e^{rt}$.

We know $p(30) = 2p_0$. So $2p_0 = p_0 e^{30r}$

So $2 = e^{30r}$. Then $30r = \ln(2)$.

So $\boxed{r = \frac{\ln(2)}{30}}$

b) what about in N days?

$p(N) = 2p_0$, so $2p_0 = p_0 e^{Nr}$, so

$2 = e^{Nr}$, so $Nr = \ln|2|$

so $\boxed{r = \frac{\ln(2)}{N}}$ ← where the units are /day.

SECTION 1.3

(12) $y_1(t) = \dfrac{1}{t^2} \Rightarrow y_1' = \dfrac{-2}{t^3}$ and $y_1'' = \dfrac{6}{t^4}$

So

$$t^2 y_1'' + 5t y_1' + 4y_1 = t^2\left(\frac{6}{t^4}\right) + 5t\left(-\frac{2}{t^3}\right) + 4\left(\frac{1}{t^2}\right)$$

$$= \frac{6-10+4}{t^2} = \frac{0}{t^2} = 0$$, so y_1 is a solution.

$y_2(t) = \dfrac{\ln(t)}{t^2}$. Then $y_2' = \dfrac{-2\ln t}{t^3} + \dfrac{1}{t^3} = \dfrac{1-2\ln t}{t^3}$

$$y_2'' = \frac{(1-2\ln t)(-3)}{t^4} - \frac{2}{t^4} = \frac{6\ln t - 5}{t^4}$$

So

$$t^2 y_2'' + 5t y_2' + 4y_2 = \frac{t^2(6\ln t - 5)}{t^4} + \frac{5t(1-2\ln t)}{t^3} + \frac{4\ln t}{t^2}$$

$$= \frac{1}{t^2}\left(6\ln t - 5 + 5(1-2\ln t) + 4\ln t\right)$$

$$= \frac{1}{t^2}\left(6\ln t - 5 + 5 - 10\ln t + 4\ln t\right)$$

$$= \frac{1}{t^2}(0) = 0.$$

So y_2 is also a solution.

Section 2.1

(15) $ty' + 2y = t^2 - t + 1, \quad y(1) = 1/2, \quad t > 0.$

$y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$. Then $p(t) = \frac{2}{t}$ and $g(t) = t - 1 + 1/t$

so $\mu(t) = \exp\int p(t)\,dt = \exp\int \frac{2}{t}\,dt = \exp(2\ln|t|) = t^2$. (since $t>0$, $|t|=t$)

Then $t^2 y' + 2ty = t^3 - t^2 + t$ and

$$t^2 y' + 2ty = \frac{d}{dt}(t^2 y) \quad \text{so}$$

$$t^2 y + C = \int (t^3 - t^2 + t)\,dt = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}$$

So $y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + C/t^2$

So $y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$.

Since $y(1) = 1/2$, then $\frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = 1/2$

$$C = -\frac{1}{4} + 1/3 = \frac{1}{12}$$

So $y(t) = \frac{t^2}{4} - \frac{t}{3} + 1/2 + \frac{1}{12t^2}$

(17) $y' - 2y = e^{2t}, \quad y(0) = 2$

$\mu(t)y' - 2\mu(t)y = \mu(t)e^{2t}$

$d(\mu(t)y) = \mu(t)y' + \mu'(t)y = \mu(t)y' - 2\mu(t)y$

if $\mu'(t) = -2\mu(t)$, so $\mu(t) = e^{-2t}$

Then $e^{-2t}y' - 2e^{-2t}y = \frac{d}{dt}\left[e^{-2t}y\right] = e^{-2t}e^{2t} = 1$

So $e^{-2t}y = t + C$

$y = e^{2t}t + Ce^{2t}$.

Since $y(0) = 2$, then $C = 2$.

So $y(t) = te^{2t} + 2e^{2t} = (t+2)e^{2t}$

33 a, λ are constants s.t $a, \lambda > 0$ and $b \in \mathbb{R}$

$y' + ay = be^{-\lambda t}$.

$\mu(t)y' + a\mu(t)y = \frac{d}{dt}\left[\mu(t)y\right]$ if $\mu'(t) = a\mu(t)$,

so $\mu(t) = e^{at}$.

Then $\frac{d}{dt}\left[e^{at}y\right] = be^{(a-\lambda)t}$.

If $a \neq \lambda$, then $e^{at}y = \frac{b}{(a-\lambda)}e^{(a-\lambda)t} + C$

So $y = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}$.

Since $\lambda > 0$ as $t \to \infty$ $e^{-\lambda t} \to 0$. Since $a > 0$ then $e^{-at} \to 0$.

So $y \to 0$ as $t \to \infty$.

If $a = \lambda$, then $\frac{d}{dt}\left[e^{\lambda t}y\right] = b$, so $e^{\lambda t}y = bt + C$,

so $y = bte^{-\lambda t} + Ce^{-\lambda t}$.

Since $e^{-\lambda t} \to 0$ then $y \to 0$ as $t \to \infty$.

SECTION 2.2

(3) $y' + y^2 \sin x = 0$

$\frac{y'}{y^2} = -\sin x$ so $\frac{dy}{y^2} = -\sin x \, dx.$

So $-\frac{1}{y} = \cos x + C$

$\boxed{y = \frac{-1}{\cos x + C}}$ if $y \neq 0$. Another solution is $y = 0$.

(8) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

$(1+y^2)\,dy = x^2\,dx$, so

$y + \frac{y^3}{3} = \frac{x^3}{3} + C$

$3y + y^3 = x^3 + 3C$

$3y + y^2 - x^3 = c$ for some constant c.

SECTION 2.4

(3) $p(t) = \tan t$, $g(t) = \sin t$.
$\sin(t)$ is continuous everywhere.
$\tan(t)$ is discontinuous at $(2k+1)\frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers).
Since $y(\pi) = 0$ and $\tan$ is discontinuous at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, then $\frac{\pi}{2} < t < \frac{3\pi}{2}$. In $(\frac{\pi}{2}, \frac{3\pi}{2})$, tan and sine are continuous.

So the interval is $\boxed{(\frac{\pi}{2}, \frac{3\pi}{2})}$

(14) $y' = 2ty^2$, $y(0) = y_0$

$\frac{dy}{y^2} = 2t\,dt$, so $-y^{-1} = t^2 + c$

$$\frac{1}{y} = -(t^2+c) \Rightarrow \boxed{y = \frac{-1}{t^2+c}}$$

Since $y(0) = y_0$, then $y_0 = \frac{-1}{c}$, so $c = \frac{-1}{y_0}$.

If $y_0 = 0$ then no such c exists, so if a solution exists it is not unique. A solution does exist since $f(t,y) = 2ty^2$ is continuous. So there are multiple solutions when $y(0) = 0$.

If $y_0 \neq 0$ then $t^2 + c \neq 0$, so $t^2 \neq \frac{1}{y_0}$ so $\frac{-1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$ if y_0 is positive.
If y_0 is negative, then $c = \frac{-1}{y_0} > 0$, so $t^2 + c > 0$ (since $t > 0$).
In that case the solution exists everywhere.

In summary: If $y_0 > 0$ then there is a unique solution for

$$\frac{-1}{\sqrt{y_0}} < t < \frac{1}{y_0}.$$

If $y_0 < 0$ then there is a unique solution everywhere.
If $y_0 = 0$ there are more solutions.

(25) $y_1' + p(t)y_1 = 0$

$y_2' + p(t)y_2 = g(t)$

$y = y_1 + y_2 \qquad y' = y_1' + y_2'$

so $y' + p(t)y = (y_1' + y_2') + p(t)(y_1 + y_2)$

$= (y_1' + p(t)y_1) + (y_2' + p(t)y_2)$

$= 0 + g(t)$

$= g(t)$.

So $y_1 + y_2$ is a solution to $y' + p(t)y = g(t)$.

(33) Let's first solve $y' + p(t)y = 0$ when $p(t) = 2$, with $y(0) = 1$

$\mu y' + 2\mu y = 0$ so $\mu(t) = e^{2t}$.

Then $\frac{d}{dt}[e^{2t}y] = 0$ so $e^{2t}y = C$

so $\boxed{y = Ce^{-2t}}$ since $y(0) = 1 \Rightarrow C = 1$.

So $y = e^{-2t}$.

so that it matches the previous eq.

Now let's solve $y' + y = 0$ with $y(1) = e^{-2(1)} = e^{-2}$.

In this case $y = Ce^{-t}$. so $e^{-2} = Ce^{-1}$ so $C = \frac{1}{e} = e^{-1}$

So $y = e^{-1}e^{-t} = e^{-t-1}$

$$\boxed{y(t) = \begin{cases} e^{-2t} & \text{if} \quad 0 \le t \le 1 \\ e^{-t-1} & \text{if} \quad t > 1. \end{cases}}$$