HOMEWORK 2 SOLUTIONS

$\boxed{2.4}$

(28) $t^2 y' + 2ty - y^3 = 0, \quad t > 0.$ So $y' + \frac{2}{t} y - \frac{y^3}{t^2} = 0$

Let $v = y^{1-3}$. Then $v' = (1-3) y^{-3} y'$

$$v' = (1-3)\frac{y'}{y^3} = (-2)\frac{\left(\frac{y^3}{t^2} - \frac{2}{t} y\right)}{y^3} = \frac{-2}{t^2} + \frac{4}{t y^2}$$

But $v = \frac{1}{y^2}$ so $\frac{4}{t y^2} = \frac{4v}{t}$.

So $v' = \frac{4v}{t} - \frac{2}{t^2}$, so $v' - \frac{4}{t} v = -\frac{2}{t^2}$.

Let $p(t) = \frac{-4}{t}$ and $g(t) = \frac{-2}{t^2}$.

Then $\mu(t) = \exp\left(\int \frac{-4}{t}\, dt\right) = t^{-4}$.

So $\frac{d}{dt}\left[\mu(t) v\right] = \frac{-2}{t^6}$. $\left(\text{because } \frac{-2}{t^2}(t^{-4}) = \frac{-2}{t^6}\right)$

So $\mu(t) v = \int -2t^{-6}\, dt = \frac{-2}{-5} t^{-5} + C = \frac{2}{5t^5} + C$

So $v = \frac{2t^4}{5t^5} + Ct^4 = \frac{2}{5t} + Ct^4$.

Then $v(t) = \left(\frac{1}{y(t)}\right)^2$ so

$$y(t) = \pm\sqrt{\frac{1}{v(t)}} = \pm\sqrt{\frac{1}{\frac{2}{5t} + Ct^4}}$$

$$y(t) = \pm\sqrt{\frac{5t}{2 + 5Ct^5}} = \pm\sqrt{\frac{5t}{2 + Kt}}.$$

So $y(t) = A\sqrt{\frac{5t}{2+Kt}}$ where A and K are constants.

$\boxed{2.3}$

② 120 L of pure water.
x g/L of salt enters at $r = 2$ L/min
Well-stirred leaves at same rate.
Find $S(t)$ in terms of x where t is in minutes.
$\to$ amount of salt

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$

$$= rx - r\frac{S}{120} = r\left(x - \frac{S}{120}\right) =$$

$$\frac{dS}{x - \frac{S}{120}} = r \quad \text{so} \quad (120)\left(-\ln\left|x - \frac{S}{120}\right|\right) = rt + C$$

$$\frac{S}{120} - x = Ce^{-rt/120} \quad (\text{different constant } C)$$

$$S = 120x + Ce^{-rt/120} \quad (\text{different } C)$$

$S(0) = 0$, ~~because~~ because the water is pure, so

$$120x + C = 0 \quad \text{so} \quad C = -120x.$$

$$S(t) = 120x\left(1 - e^{-\frac{rt}{120}}\right).$$

as $t \to \infty$ $e^{-rt/120} \to 0$ so $S(t) \to 120x$.

$120x$ grams of salt is the limiting value.

④ Let $Q(t)$ = Amount of water in the tank after t minutes
$S(t)$ = Amount of salt after t minutes.

$Q(0) = 200$ gal
$S(0) = 100$ lb.

$$\frac{dQ}{dt} = 3\,\text{gal/min} - 2\,\frac{\text{gal}}{\text{min}} = 1\,\frac{\text{gal}}{\text{min}} \qquad Q = t + C$$

$Q(0) = 200$ so $C = 200$

Then $Q(t) = t + 200$ (as long as $t \le 300$).

If $t > 300$, then $Q(t) = 500$.

rate out

$$\frac{dS}{dt} = 3(1) - \frac{2S}{Q} = 3 - \frac{2S}{t+200} \quad \text{(as long as } t \le 300\text{)}$$

rate in, lb of salt per gallon, amount of salt, amount of water

$$S' = 3 - \frac{2S}{t+200}. \quad \text{Then } S' + \frac{2}{t+200} = 3$$

$$p(t) = \frac{2}{t+200} \quad \text{and} \quad g(t) = 3$$

$$\mu(t) = \exp\left(\int \frac{2}{t+200}\,dt\right) = \exp\left(2\ln(t+200)\right) = (t+200)^2.$$

$$\text{Then } (t+200)^2 S = \int 3(t+200)^2\,dt$$

$$= \frac{3(t+200)^3}{3} + C = (t+200)^3 + C$$

$$\text{So } S = (t+200) + \frac{C}{t+200}.$$

$$S(0) = 100 \quad \text{so} \quad 200 + \frac{C}{200} = 100 \qquad \frac{C}{200} = -100 \quad \text{so } C = -20000.$$

$$\text{Then } \boxed{S(t) = t + 200 - \frac{20000}{t+200}} \quad \text{(as long as } t \le 300\text{)}$$

The concentration is $\frac{S(t)}{Q(t)}$. The question asks for $\frac{S(300)}{Q(300)}$

$$\text{so } \frac{500 - \frac{20000}{500}}{500} = 1 - \frac{40}{500} = 1 - \frac{4}{50} = \boxed{0.92}$$

If it had infinite capacity then we want

$$\lim_{t\to\infty} \frac{S}{Q}(t) = \lim_{t\to\infty} \frac{(t+200) - \frac{20000}{t+200}}{t+200} = \lim_{t\to\infty} 1 - \frac{20000}{(t+200)^2} = \boxed{1}.$$

starts with 0.

(8) k dollars a year at annual rate r. continuous deposit and continuous compounding

a) $S(0)=0$.

$$\frac{dS}{dt} = k + rS, \text{ so } \frac{dS}{rS+k} = dt \text{ so}$$

$$\frac{1}{r}\ln\left|S+\frac{k}{r}\right| = t + C$$

$$\ln\left|S+\frac{k}{r}\right| = rt + C \quad \text{so} \quad S(t) = Ce^{rt} - \frac{k}{r}$$

Since $S(0)=0$ then $C=\frac{k}{r}$, so $\boxed{S(t)=\frac{k}{r}(e^{rt}-1)}$

b) $r = 7.5\%$, so $r = .075$. Find k such that $S(40) = 1000000$.

$$1000000 = S(40) = \frac{k}{.075}\left(e^{.075t}-1\right) \quad \text{so}$$

$$75000 = k\left(e^{.075(40)}-1\right) = k\left(e^{3}-1\right)$$

$$k = \frac{75000}{e^{3}-1} \approx \boxed{3929.677}$$

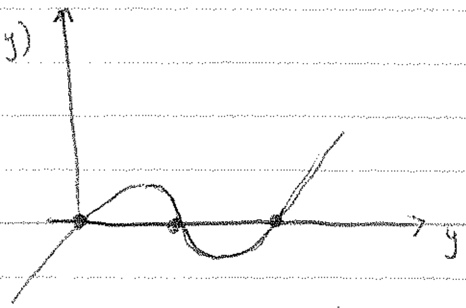
c) If $k = 2000$. what is r so that $S(40)=1000000$.

$$1000000 = \frac{2000}{r}\left(e^{40r}-1\right) \quad \text{so} \quad 500r = e^{40r}-1$$

We use Mathematica to solve this: $r \approx 0.0977$.

$\boxed{2.5}$

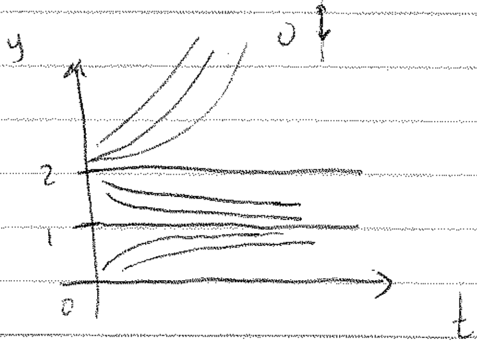
(3) $y' = y(y-1)(y-2)$, $y_0 \geq 0$



The equilibrium points are 0, 1 and 2.

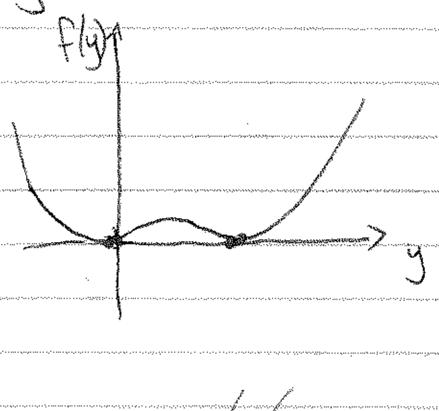
Phaseline:

So $y=1$ is stable while $y=0$ and $y=2$ are unstable.



Sketch of solutions

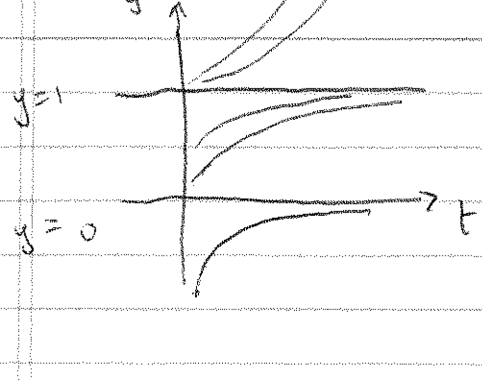
(13) $y' = y^2(1-y)^2$, $-\infty < y_0 < \infty$



The equilibrium points are 0 and 1.

Phase line:

Both are semistable.



Sketch of solutions.

(15) $\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$

a) $y_0 = K/3$. Find τ at which the pop. doubled.
$r = 0.025$

As done in the book (equation (11))

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Since $y_0 = K/3$ and $r = .025$ then

$$y(t) = \frac{K^2/3}{\frac{K}{3} + \left(\frac{2K}{3}\right)e^{-.025t}} = \frac{K}{1 + 2e^{-0.025t}}$$

$$y(\tau) = \frac{2K}{3} \quad \text{so} \quad \frac{2K}{3} = \frac{K}{1 + 2e^{-0.025\tau}}$$

$$2 + 4e^{-0.025\tau} = 3, \quad \text{so}$$

$$e^{-0.025\tau} = \frac{1}{4} \quad \text{so} \quad -0.025\tau = \ln\left(\frac{1}{4}\right)$$

$$\tau = \frac{1}{.025}\ln(4) = 55.4518 \text{ years}.$$

b) $y_0/K = \alpha$. Find T s.t $\frac{y(T)}{K} = \beta$.

$$\beta K = y(T) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rT}} = \frac{\alpha K^2}{\alpha K + (1-\alpha)Ke^{-rT}} = \frac{\alpha \cancel{K}}{\alpha + (1-\alpha)e^{-rT}}$$

So $\beta = \frac{\alpha}{\alpha + (1-\alpha)e^{-rT}}$. So $(1-\alpha)\beta e^{-rT} = \alpha(1-\beta)$

$$e^{-rT} = \frac{\alpha}{1-\alpha}\frac{(1-\beta)}{\beta}$$

$$T = -\frac{1}{r}\ln\left(\frac{\alpha}{1-\alpha} \cdot \frac{1-\beta}{\beta}\right).$$ As $\alpha \to 0$ $\ln\left(\frac{\alpha}{1-\alpha} \cdot \frac{1-\beta}{\beta}\right) \to -\infty$, so $T \to \infty$

As $\beta \to 1$ $\ln\left(\frac{\alpha}{1-\alpha} \cdot \frac{1-\beta}{\beta}\right) \to -\infty$, so $T \to \infty$.

$\alpha = .1$, $\beta = .9$, $r = .025$ then

$$T = \frac{-1}{0.025}\ln\left(\left(\frac{.1}{.9}\right)^2\right) = \boxed{175.778} \text{ years}$$

$\boxed{2.6}$

(2) $(2x+4y)+(2x-2y)y'=0$.

So $M(x,y)=2x+4y$ and $N(x,y)=2x-2y$

$M_y(x,y)=4$ and $N_x(x,y)=2$

Since $4\neq 2$, it is not exact.

(14) $(9x^2+y-1)-(4y-x)y'=0$, $y(1)=0$

$M(x,y)=9x^2+y-1$, $N(x,y)=x-4y$

$M_y=1$ $N_x=1$

So it is exact.

$\psi_x(x,y)=9x^2+y-1$

$\psi_y(x,y)=x-4y$

$\psi(x,y)=\int \psi_x(x,y)dx=3x^3+h(y)+(y-1)x$

$\psi_y(x,y)=h'(y)+x$

So $h'(y)=-4y$. So $h(y)=-2y^2$.

So $\psi(x,y)=3x^3+(y-1)x-2y^2$

So $\boxed{3x^3+(y-1)x-2y^2=c}$

Since $y(1)=0$ then $3(1)^3+(0-1)(1)-2(0)^2=c$

so $c=2$

Then $3x^3+(y-1)x-2y^2=2$

$2y^2-yx+(x-3x^3+2)=0$ Then $y=\frac{x\pm\sqrt{x^2-8x+24x^3-16}}{4}$

$$y(x) = \frac{x \pm \sqrt{24x^3 + x^2 - 8x - 16}}{2}$$

Since $y(1) = 0$ then $y(x) = \dfrac{x - \sqrt{24x^3 + x^2 - 8x - 16}}{4}$.

The equation is valid as long as

$$24x^3 + x^2 - 8x - 16 \geq 0.$$

$24x^3 + x^2 - 8x - 16 = 0$ implies $x \approx 0.9846$.

So the equation is valid as long as $x \geq 0.9846$.

(24) Suppose $\dfrac{N_x - M_y}{xM - yN} = R(xy)$.

Let $\mu(xy)$ be a function of xy. To be the integrating factor it must satisfy

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

so $(N_x - M_y)\mu = M\mu_y - N\mu_x$.

$\mu_x = y\,\mu'(xy)$ by Chain Rule
$\mu_y = x\,\mu'(xy)$

so $(N_x - M_y)\mu = Mx\mu' - Ny\mu' = \mu'(Mx - Ny)$

so $\left(\dfrac{N_x - M_y}{Mx - Ny}\right)\mu = \mu'$ so $R\mu = \mu'$

so $\dfrac{\mu'}{\mu} = R$ so $\mu = \exp\int R(t)\,dt$ where $t = xy$.

Indeed if $\mu(xy) = \exp\int R(xy)\,d(xy)$

then $\mu_x = y\,\mu'(xy) = y\,R(xy)\exp\left(\int R(xy)\,d(xy)\right) = y\,R(xy)\,\mu(xy)$
and $\mu_y = x\,R(xy)\,\mu(xy)$

so $M\mu_y - N\mu_x = x\,M(x,y)\,R(xy)\,\mu(xy) - y\,N(x,y)\,R(xy)\,\mu(xy)$
$= R(xy)\,\mu(xy)\,(xM - yN)$

and $N_x - M_y = R(xy)(xM - yN)$, so

$M\mu_y - N\mu_x = (N_x - M_y)\,\mu(xy)$. So μ is an integrating factor.

$\boxtimes$

(28) $y + (2xy - e^{-2y})y' = 0$

$N_x = 2y$
$M_y = 1$ Since $M_y \neq N_x$, it is not exact

However $\frac{N_x - M_y}{\cancel{xy}\, y} = \frac{2y-1}{y}$ is a function of y.

Let $Q(y) = 2y - 1$. Then $\mu(y) = \exp \int Q(y)\,dy$ is an integrating factor. Then $\mu(y) = \exp\left(\int \frac{2y-1}{y}\,dy\right)$

$$= \exp\left(\int (2 - \tfrac{1}{y})\,dy\right) = \exp\left(2y - \ln(y)\right) = \frac{e^{2y}}{y}.$$

Then $\frac{e^{2y}}{y} y + \frac{e^{2y}}{y}(2xy - e^{-2y})y' = 0$

so $e^{2y} + (2xe^{2y} - \frac{1}{y})y' = 0.$

Now let $M(x,y) = e^{2y}$ and $N(x,y) = 2xe^{2y} - 1/y$.

$M_y = 2e^{2y}$ and $N_x = 2e^{2y}$ so $M_x = N_y$, so the DFQ is <u>exact</u>

Then we want $\Psi_x = e^{2y}$ and $\Psi_y = 2xe^{2y} - 1/y$

$\Psi_x = e^{2y} \Rightarrow \Psi = xe^{2y} + h(y)$
so $\Psi_y = 2xe^{2y} + h'(y)$.

So we want $h'(y) = -1/y$. So $h(y) = -\ln(y)$.

So $\Psi(x,y) = xe^{2y} - \ln(y)$.

Then the DFQ is solved by the implicit equation

$$\boxed{xe^{2y} - \ln(y) = C}$$