Homework 3 Solutions

$\boxed{3.1}$

(11) $6y'' - 5y' + y = 0$. with $y(0) = 4$, $y'(0) = 0$

Suppose $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2 e^{rt}$

so $(6r^2 - 5r + 1)e^{rt} = 0$

$6r^2 - 5r + 1 = 0$ so $r = \frac{5 \pm \sqrt{25-24}}{12} = \frac{5 \pm 1}{12}$

so $r = \frac{1}{2}$ and $r = 1/3$.

Then $y(t) = c_1 e^{t/2} + c_2 e^{t/3}$

$y'(t) = \frac{1}{2} c_1 e^{t/2} + \frac{1}{3} c_2 e^{t/3}$

$y(0) = c_1 + c_2 = 4$

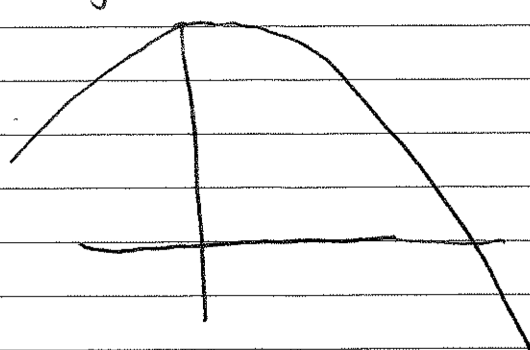
$y'(0) = \frac{1}{2}c_1 + \frac{1}{3}c_2 = 0$

$3c_1 + 2c_2 = 0$ so $c_1 = -\frac{2}{3}c_2$

so $4 = c_1 + c_2 = -\frac{2}{3}c_2 + c_2 = \frac{1}{3}c_2$. so $c_2 = 12$

and $c_1 = -8$

Hence $y(t) = -8e^{t/2} + 12e^{t/3}$.



It goes to $-\infty$ as $t \to \infty$

(12) $y'' + 3y' = 0$, $y(0) = -2$, $y'(0) = 3$.

$y = e^{rt}$ so $y'' + 3y' = (r^2 + 3r)e^{rt} = 0$

So $r^2 + 3r = 0$ so $r = 0$ and $r = -3$.

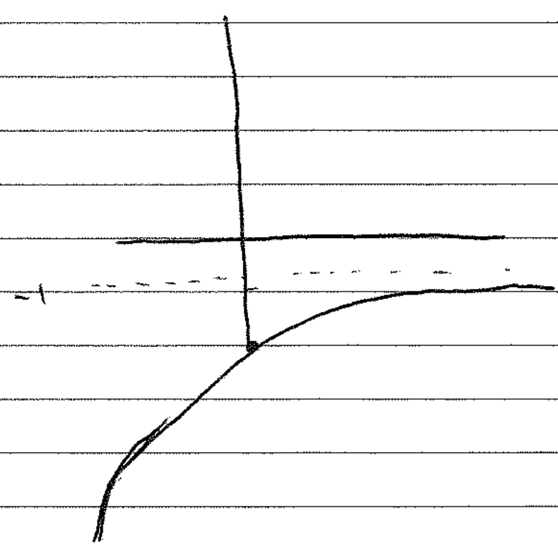
Then $y(t) = c_1 e^{0t} + c_2 e^{-3t}$
$= c_1 + c_2 e^{-3t}$.

$y(0) = -2$ so $c_1 + c_2 = -2$
$y'(0) = 3$ so $-3c_2 = 3$

Since $-3c_2 = +3$ so $c_2 = -1$.

Then $c_1 = -2 - (-1)$
$= -1$.

$$\boxed{y(t) = -1 - e^{-3t}}$$



$t \to \infty$.
$y \to -1$

$\boxed{3.2}$

(11) $(x-3)y'' + xy' + (\ln|x|)y = 0 \qquad y(1)=0,\ y'(1)=1$

so $y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$

So $x_0 = 1$.

$x/x-3$ is continuous everywhere except $x \neq 3$

$\ln|x|$ is continuous in $(-\infty, 0)$ or $(0, \infty)$.

So we have several candidates: $(-\infty,0), (0,3), (3,\infty)$. Since $1 \in (0,3)$. Then $\boxed{I = (0,3)}$

(19)
$$W(u,v) = \begin{vmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{vmatrix}$$

$u = 2f - g$
$u' = 2f' - g'$
$v = f + 2g$
$v' = f' + 2g'$

$= uv' - u'v = (2f-g)v' - u'(f+2g)$

$= (2f-g)(f'+2g') - (2f'-g')(f+2g)$

$= 2ff' + 4fg' - gf' - 2gg' - (2ff' + 4f'g - g'f - 2gg')$

$= 2ff' + 4fg' - f'g - 2gg' - 2ff' - 4f'g + fg' + 2gg'$

$= 5fg' - 5f'g = 5(fg' - f'g) = 5W(f,g)$.

(23) $y'' + 4y' + 3y = 0$, $t_0 = 1$. We want the fundamental solutions;

Let $y = e^{rt}$. Then $r^2 + 4r + 3 = 0$. So $(r+3)(r+1) = 0$

So $r = -1$ or $r = -3$.

Then $y_1 = c_1 e^{-t} + c_2 e^{-3t}$.

$$y_1(t) = c_1 e^{-t} + c_2 e^{-3t}$$

with $y_1(1) = 1$ and $y_1'(1) = 0$

$$c_1 e^{-1} + c_2 e^{-3} = 1$$
$$-c_1 e^{-1} - 3c_2 e^{-3} = 0$$

So $-2c_2 e^{-3} = 1$ so $c_2 = -\frac{e^3}{2}$

$$c_1 e^{-1} + \left(-\frac{e^3}{2} e^{-3}\right) = 1$$
$$c_1 e^{-1} = 1 + 1/2 = 3/2$$

$$\boxed{c_1 = \frac{3e}{2}}$$

Then $\boxed{y_1(t) = \frac{3e}{2} e^{-t} - \frac{e^3}{2} e^{-3t}}$

$$y_1(t) = \frac{3}{2} e^{1-t} - \frac{1}{2} e^{3-3t}$$

Similarly $y_2(1) = 0$ and $y_2'(1) = 1$

so $c_1 e^{-1} + c_2 e^{-3} = 0$
$$-c_1 e^{-1} - 3c_2 e^{-3} = 1$$

so $-2c_2 e^{-3} = +1$

so $c_2 = -e^3/2$

$c_1 e^{-1} = +\frac{e^3}{2} e^{-3} = +1/2$ so $c_1 = \frac{+e}{2}$

$$\boxed{y_2(t) = \frac{+e}{2} e^{-t} - \frac{e^3}{2} e^{-3t}}$$

$$= +\frac{1}{2} e^{1-t} - \frac{1}{2} e^{3-3t}$$

$\boxed{27}$ $(1-x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi$

$y_1(x) = x \quad y_1' = 1 \quad y_1'' = 0$

$\therefore \ (1 - x\cot x)(0) - x(1) + x = 0 \ \checkmark$

because $0 - x + x = 0$

$y_2(x) = \sin x. \qquad y_2' = \cos x \ , \quad y_2'' = -\sin x$

$(1 - x\cot x)(-\sin x) - x\cos x + \sin x$

$= -\sin x + x\cot x \sin x - x\cos x + \sin x$

$= x\left(\frac{\cos x}{\sin x}\right)\sin x - x\cos x \ = x\cos x - x\cos x = 0.$

So y_1 and y_2 are solutions to the DFQ.

Let's check if they are fundemental solutions:

$y_1 y_2' - y_1' y_2 = \ x\cos x - (1)\sin x = x\cos x - \sin x.$

Let $x = \pi/2$, then $x\cos x - \sin x = -\sin\left(\frac{\pi}{2}\right) = -1 \neq 0.$

Therefore y_1 and y_2 are fundemental solutions.

$\boxed{35}$ $t^2 y'' - 2y' + (3+t)y = 0$ has y_1 and y_2 as fund. sols and $W(y_1, y_2)(2) = 3.$

$W(y_1, y_2)(4) = y_1(4)y_2'(4) - y_1'(4)y_2(4)$

By <u>abel's theorem</u> $W(y_1, y_2)(t) = c\exp\left[-\int p(t)\,dt\right]$

so $W(y_1, y_2)(t) = c \ \exp\left[-\int -\frac{2}{t^2}\,dt\right]$

$= c \ \exp\left[-\frac{2}{t}\right] = ce^{-2/t}$

$W(y_1, y_2)(2) = 3 = ce^{-2/2} = ce^{-1}$ Therefore $c = 3e$

Then $W(y_1, y_2)(4) = ce^{-2/4} = 3e \cdot e^{-1/2} = \boxed{3\sqrt{e}}$

39 CASE 1: y_1 and y_2 have maxima at the same point in I.
Let's call this point t_M.
Then $y_1'(t_M) = 0$ and $y_2'(t_M) = 0$.

So $W(y_1, y_2)_{(t_M)} = -y_1'(t_M)\, y_2(t_M) + y_1(t_M)\, y_2'(t_M)$

$$= 0 - 0 = 0.$$

Since $W = 0$ then y_1 and y_2 are not fundamental solutions.

CASE 2: The argument if y_1, y_2 have the same minima is the same as the key is that at that point both y_1' and y_2' are zero.

$\boxed{3.3}$

① $e^{1+2i} = e \cdot e^{2i} = e\left(\cos(2) + i\sin(2)\right)$.
$= e\cos(2) + i\left(e\sin(2)\right)$.

⑤ $2^{1-i} = e^{\ln(2)(1-i)} = e^{\ln(2) - \ln(2)i} = e^{\ln(2)}\, e^{-\ln(2)i}$
$= 2 \cdot e^{-\ln(2)i} = 2\left(\cos(-\ln(2)) + i\sin(-\ln(2))\right)$
$= 2\cos(\ln(2)) - i\left(2\sin(\ln(2))\right)$.

⑪ $y'' + 6y' + 13y = 0$. Let $y = e^{rt}$.

Then $r^2 + 6r + 13 = 0$

So
$$r = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$

Then $y = e^{(-3+2i)t}$ is a solution.

$$y = e^{-3t}\left(\cos(2t) + i\sin(2t)\right) = e^{-3t}\cos(2t) + i\left(e^{-3t}\sin(2t)\right).$$

So $e^{-3t}\cos(2t)$ and $e^{-3t}\sin(2t)$ are solutions to the DEQ

$y_1 = e^{-3t}\cos(2t)$ and $y_2 = \sin(2t)e^{-3t}$

$W(y_1, y_2) = 2\cos(2t)\cos(2t)e^{-6t} - \left(-2\sin(2t)\right)\left(\sin(2t)\right)e^{-6t} + \cancel{e^{-6t}\cos(2t)(-3\sin} - \cancel{e^{-6t}\sin(2t)(-3\cos(2t))}$
$= 2e^{-6t}\left(\cos^2(2t) + \sin^2(2t)\right) = 2e^{-6t} \neq 0$.

So y_1 and y_2 are fundamental solutions.

So the general solution is $\boxed{y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)}$

(19) $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.

$r^2 - 2r + 5 = 0 \Rightarrow r = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$y = e^{(1+2i)t}$ so $y = e^t(\cos(2t) + i\sin(2t))$.

So $y_1 = e^t\cos(2t)$ and $y_2 = e^t\sin(2t)$

Then $W = e^{2t}(2\cos(2t)\cos(2t) + \cos(2t)\sin(2t))$

$- e^{2t}(-2\sin(2t)\sin(2t) + \cos(2t)\sin(2t))$

$= 2e^{2t}(\cos^2(2t) + \sin^2(2t)) = 2e^{2t} \neq 0$.

Therefore $y = c_1 e^t\cos(2t) + c_2 e^t\sin(2t)$.

$y(\pi/2) = c_1 e^{\pi/2}\cos(\pi) + c_2 e^{\pi/2}\sin(\pi) = -c_1 e^{\pi/2}$

Since $y(\pi/2) = 0$ then $c_1 = 0$.

$y = c_2 e^t \sin(2t)$. $y'(t) = c_2 e^t(2\cos(2t))$

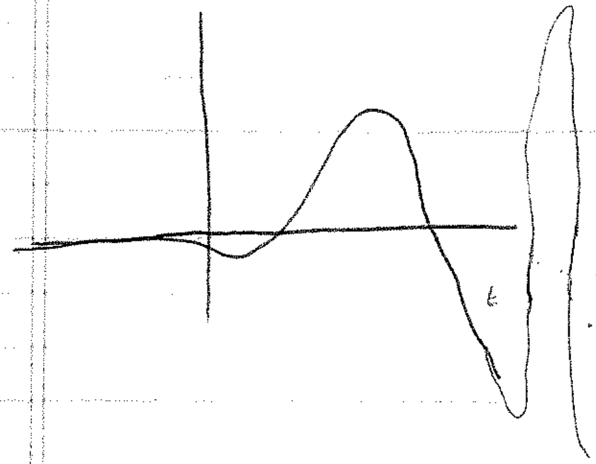
$+ c_2 \sin(2t) e^t$

$= c_2 e^t(2\cos(2t) + \sin(2t))$

Then $y'(\pi/2) = c_2 e^{\pi/2}(2\cos\pi + \sin\pi) = -2c_2 e^{\pi/2}$.

Since $y'(\pi/2) = 2$, then $-2c_2 e^{\pi/2} = 2$ so $c_2 = -e^{-\pi/2}$

$$\boxed{y(t) = -e^{t-\pi/2}\sin(2t)}$$



It oscillates enormously because $\sin(2t)$ goes from 1 to -1 and back.
So the function is unbounded in very chaotic fashion.

3.4

2) $9y'' + 6y' + y = 0$

$9r^2 + 6r + 1 = 0$ so $(3r+1)^2 = 0$ so $r = -1/3$.

Then the general solution is
$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}.$$

13) $9y'' + 6y' + 82y = 0$, $y(0) = -1$, $y'(0) = 2$.

$9r^2 + 6r + 82 = 0$

$$r = \frac{-6 \pm \sqrt{36 - 4(9)(82)}}{18} = \frac{-6 \pm \sqrt{36(1-82)}}{18}$$

$$= \frac{-6 \pm 6(9i)}{18} = -\frac{1}{3} \pm 3i$$

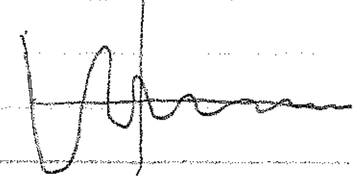
Then $y(t) = e^{(-\frac{1}{3}+3i)t} = e^{-t/3}(\cos(3t) + i\sin(3t))$.

so $y(t) = c_1 e^{-t/3}\cos(3t) + c_2 e^{-t/3}\sin(3t)$.

$y(0) = -1$ so $-1 = c_1 e^{-0/3}\cos(0) + 0 = c_1$

so $\boxed{c_1 = -1}$

$$y'(t) = -\frac{1}{3}c_1 e^{-t/3}\cos(3t) - 3c_1 e^{-t/3}\sin(3t) - \frac{1}{3}c_2 e^{-t/3}\sin(3t) + 3c_2 e^{-t/3}\cos(3t)$$



$y'(0) = -\frac{1}{3}(-1) + 3c_2 = 2$

$3c_2 = 2 - \frac{1}{3} = \frac{5}{3}$ so $\boxed{c_2 = \frac{5}{9}}$

$$\boxed{y(t) = -e^{-t/3}\cos(3t) + \frac{5}{9}e^{-t/3}\sin(3t)}$$

Since $e^{-t/3} \to 0$ as $t \to \infty$ then $y \to 0$ as $t \to \infty$

(14) $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

$r^2 + 4r + 4 = 0$ so $(r+2)^2 = 0$ so $r = -2$.

Then $\boxed{y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.}$

$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$y(-1) = c_1 e^2 - c_2 e^2 = 2$

$y'(-1) = -2c_1 e^2 + c_2 e^2 + 2c_2 e^2 = 1$

$e^2(c_1 - c_2) = 2 \Rightarrow e^2(2c_1 - 2c_2) = 4$

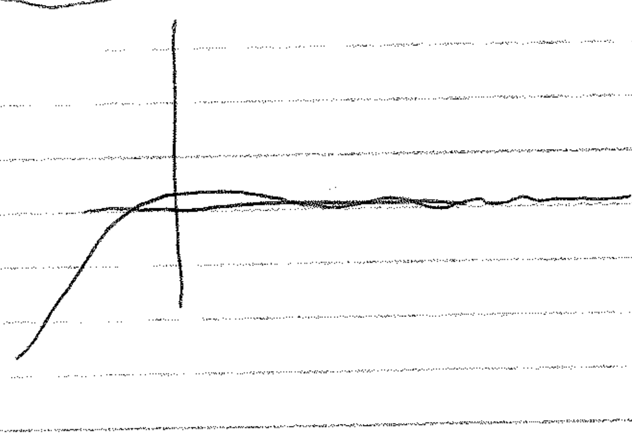
$e^2(-2c_1 + 3c_2) = 1 \qquad e^2(-2c_1 + 3c_2) = 1$

$e^2(c_2) = 5$ so $\boxed{c_2 = 5e^{-2}}$

Then $e^2(c_1 - 5e^{-2}) = 2$

$e^2 c_1 - 5 = 2 \qquad \boxed{c_1 = 7e^{-2}}$

$$\boxed{y(t) = 7e^{-2-2t} + 5te^{-2-2t}}$$



as $t \to \infty$

$y \to 0$.

(23) $t^2 y'' - 4ty' + 6y = 0$, $t > 0$; $y_1(t) = t^2$.

Let $y(t) = v(t)\, t^2$.

Then $y'(t) = v'(t)\,t^2 + 2t\,v(t)$

$y'' = v''(t)\,t^2 + 2t\,v'(t) + 2t\,v'(t) + 2v(t)$

So $t^2 y'' - 4ty' + 6y = 0$

implies

$$t^2\left(v''(t)\,t^2 + 4t\,v'(t) + 2v(t)\right) - 4t\left(v'(t)\,t^2 + 2t\,v(t)\right)$$
$$+\ 6v(t)\,t^2 = 0$$

so

$$v''t^4 + 4\cancel{t^3v'} + 2\cancel{t^2v} - 4\cancel{t^3v'} - 8\cancel{t^2}v + 6\cancel{t^2}v = 0$$

so $v''t^4 = 0$.

so $v'' = 0$. Then $v' = c$

so $v = ct + d$.

We can pick $v = t$, so $y = t^3$.

Then $y_1 = t^2$ and $y_2 = t^3$.

(25) $t^2 y'' + 3ty' + y = 0$, $t > 0$; $y_1(t) = t^{-1}$.

$y_2(t) = vt^{-1}$ so $y_2' = v't^{-1} - vt^{-2}$

$y_2'' = v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3}$

$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$.

Then $t^2 y_2'' + 3t y_2' + y_2 = 0$

implies

$$t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) + vt^{-1} = 0$$

$$tv'' - 2v' + 2vt^{-1} + 3v' - 3vt^{-1} + vt^{-1} = 0$$

$$tv'' + v' = 0.$$

$tv'' + v' = 0$ so

$v'' = -\frac{v'}{t}$ so $\frac{v''}{v'} = -\frac{1}{t}$

so $\ln|v'| = -\ln t$ so $v' = t^{-1}$

so $v = \int t^{-1}\,dt = \ln t$. ($t > 0$ so no need for $|\,|$).

Then $v(t) = \ln t$, so $y_2(t) = \ln t\,(t^{-1})$

$$\boxed{y_2(t) = \frac{\ln t}{t}}$$

(Note: choosing different constants yields slightly different $y_2(t)$, for example $v' = Ct^{-1}$, so

$v = C\ln t + D$ so

$y_2(t) = \frac{C\ln t}{t} + \frac{D}{t}$ for any constants $C \neq 0$ and D.

Similarly in (23) the choice of constants can lead to $y_2(t) = ct^3 + dt^2$)