HOMEWORK 4 SOLUTIONS

3.5

① $y'' - 2y' - 3y = 3e^{2t}$

Let's solve the homogeneous equation first:

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0 \quad \text{so} \quad r = 3 \text{ and } r = -1.$$

So $y(t) = c_1 e^{3t} + c_2 e^{-t}$ is the gen. sol. of $y'' - 2y' - 3y = 0$

Now let's find a particular solution to ①.

Let $y = Ae^{2t}$. Then $y' = 2Ae^{2t}$, $y'' = 4Ae^{2t}$

so ① becomes $4Ae^{2t} - 2(2Ae^{2t}) - 3Ae^{2t} = 3e^{2t}$

So $4A - 4A - 3A = 3$

$$\boxed{A = -1}$$

So $y(t) = -e^{2t}$ is a solution to ①.

Therefore $\boxed{y(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t}}$ is the general solution.

② $y'' + 2y' + 5y = 3\sin 2t$.

Let's solve the homogeneous equation:

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

So $y = e^{rt} = e^{(-1+2i)t} = e^{-t} e^{2it} = e^{-t}(\cos(2t) + i\sin(2t))$

So $y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$ is a general solution to $y'' + 2y' + 5y = 0$

Let's find a particular solution:

Suppose $y = A\sin(2t) + B\cos(2t)$.

Then $y' = 2A\cos(2t) - 2B\sin(2t)$, $y'' = -4A\sin(2t) - 4B\cos(2t)$

So $y'' + 2y' + 5y = 3\sin 2t$ implies

$$3\sin(2t) = -4A\sin(2t) - 4B\cos(2t) + 4A\cos(2t) - 4B\sin(2t) + 5A\sin(2t) + 5B\cos(2t)$$

So
$$3 = -4A - 4B + 5A = A - 4B$$
$$0 = -4B + 4A + 5B = 4A + B$$

$A = 4B + 3$ so $4(4B+3) + B = 0$

$17B + 12 = 0$ so $B = \frac{-12}{17}$

$A = 4B + 3 = \frac{-48}{17} + 3 = \frac{3}{17}$

So $y(t) = \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$

So the general solution to ② is

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$$

(13) $y'' + y' + 4y = e^t - e^{-t}$. (because $2\sinh(t) = e^t + e^{-t}$)

Then the homogeneous equation is
$$y'' + y' + 4y = 0$$

So $r^2 + r + 4 = 0$

So $r = \frac{-1 \pm \sqrt{1 - 16}}{2} = \frac{-1 \pm \sqrt{15}\,i}{2}$

$$y = e^{rt} = e^{-\frac{1}{2}t} e^{\frac{\sqrt{15}}{2}it} = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + i e^{t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

So $y(t) = c_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$, is the general solution to

$$y'' + y' + 4y = 0.$$

Now let's find a particular solution. Since $g(t) = e^t + e^{-t}$.
Let's break it into two pieces.

Let's solve $y'' + y' + 4y = e^t$

and $y'' + y' + 4y = -e^{-t}$.

First for $y'' + y' + 4y = e^t$, suppose $y = Ae^t$.

$y'' = Ae^t = y' = y$ so

$$Ae^t + Ae^t + 4Ae^t = e^t \quad \text{so} \quad \boxed{A = \frac{1}{6}}$$

Then $y(t) = \frac{1}{6}e^t$.

For $y'' + y' + 4y = -e^{-t}$, suppose $y = Ae^{-t}$.

$y'' = Ae^{-t} = y$ and $y' = -Ae^{-t}$.

$$Ae^{-t} - Ae^{-t} + 4Ae^{-t} = -e^{-t} \quad \text{so} \quad A = -1/4$$

So $y(t) = -\frac{1}{4}e^{-t}$.

Then the general solution to (13) is

$$y(t) = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

(14) $y'' - y' - 2y = \frac{e^{2t} + e^{-2t}}{2}$.

Homogeneous: $r^2 - r - 2 = 0$, so $(r-2)(r+1) = 0$,

so $r = 2$ and $r = -1$.

Therefore $y = c_1 e^{2t} + c_2 e^{-t}$ solves $y'' - y' - 2y = 0$.

Now let's solve $y'' - y' - 2y = \frac{1}{2}e^{-2t}$.

Let $y = Ae^{-2t}$, $y' = -2Ae^{-2t}$, $y'' = 4Ae^{-2t}$

So $y'' - y' - 2y = \frac{1}{2}e^{-2t}$ implies

$4Ae^{-2t} + 2Ae^{-2t} - 2Ae^{t} = \frac{1}{2}e^{-2t}$, so $A = +1/8$

So $y(t) = +\frac{1}{8}e^{-2t}$.

Now let's solve $y'' - y' - 2y = \frac{1}{2}e^{2t}$.

Since e^{2t} is a solution to the homogeneous DFQ, then let $y = Ate^{2t}$.

So $y' = Ae^{2t} + 2Ate^{2t}$

$y'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} = 4Ate^{2t} + 4Ae^{2t}$.

So $y'' - y' - 2y = \frac{1}{2}e^{2t}$ implies

$4Ate^{2t} + 4Ae^{2t} - Ae^{2t} - 2Ate^{2t} - 2Ate^{2t} = \frac{1}{2}e^{2t}$

$+3Ae^{2t} = \frac{1}{2}e^{2t}$

$A = +1/6$

so $y(t) = +\frac{1}{6}te^{2t}$.

Combining everything we get that the gen. sol. to (14) is

$$y(t) = c_1e^{2t} + c_2e^{-t} + \frac{1}{8}e^{3t} + \frac{1}{6}te^{2t}$$

(18) $y'' - 2y' - 3y = 3te^{2t}$, $y(0) = 1$, $y'(0) = 0$.

The homogeneous eq is: $y'' - 2y' - 3y = 0$ so

$r^2 - 2r - 3 = 0$ so

$(r-3)(r+1) = 0$ so $r = 3$ and $r = -1$.

Then $y = c_1e^{3t} + c_2e^{-t}$ is the gen. sol. to $y'' - 2y' - 3y = 0$.

Now $y'' - 2y' - 3y = 3te^{2t}$. Let $y = Ate^{2t} + Be^{2t}$

So $y' = 2Ate^{2t} + Ae^{2t} + 2Be^{2t}$

$y'' = 4Ae^{2t} + 4Ate^{2t} + 4Be^{2t}$

$3te^{2t} = 4Ate^{2t} + 4Ae^{2t} - 4Ate^{2t} - 2Ate^{2t} - 3Ate^{2t} + 4Be^{2t} - 4Be^{2t} - 3Be^{2t}$

So $3te^{2t} = -3Ate^{2t} + (2A-3B)e^{2t}$

so $-3A = 3$ and

$2A - 3B = 0$.

so $A = -1$ and $B = -\frac{2}{3}$

So $y(t) = -te^{2t} - \frac{2}{3}e^{2t}$.

The general solution is then

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - te^{2t} - \frac{2}{3}e^{2t}$$

Now $y(0) = 1$ and $y'(0) = 0$.

$y(0) = c_1 + c_2 - 2/3$

$y'(t) = 3c_1 e^{3t} - c_2 e^{-t} - e^{2t} - 2te^{2t} - \frac{4}{3}e^{2t}$

$y'(0) = 3c_1 - c_2 - 1 - 4/3 = 3c_1 - c_2 - 7/4$.

So $c_1 + c_2 - 2/3 = 1$ so $c_1 + c_2 = 5/3$

$3c_1 - c_2 - 7/3 = 0$ so $3c_1 - c_2 = 7/3$

Then $4c_1 = \frac{5}{3} + \frac{7}{3} = 4$

so $c_1 = 1$ Then $c_2 = 5/3 - 1 = 2/3$

So $$y(t) = e^{3t} + \frac{2}{3}e^{-t} - te^{2t} - \frac{2}{3}e^{2t}$$

(19) $y'' + 4y = 3\sin 2t$, $y(0) = 2$, $y'(0) = -1$.

Homogeneous: $r^2 + 4 = 0$ So $r = \pm 2i$

so $y = c_1\cos(2t) + c_2\sin(2t)$.

Now for the non-homogeneous let $y = A\sin(2t) + B\cos(2t)$.

Then $y' = 2A\cos(2t) - 2B\sin(2t)$

$y'' = -4A\sin(2t) - 4B\cos(2t)$

$y'' + 4y = 3\sin(2t)$

$-4A\sin(2t) - 4B\cos(2t) + 4A\sin(2t) + 4B\cos(2t) = 0.$

So it can't solve $3\sin(2t)$.

Therefore we need to multiply by t.

$y = A\sin(2t) + Bt\sin(2t) + C\cos(2t) + Dt\cos(2t)$

$y' = 2A\cos(2t) + B\sin(2t) + 2Bt\cos(2t) - 2C\sin(2t) + D\cos(2t) - 2Dt\sin(2t)$

$y'' = -4A\sin(2t) + 2B\cos(2t) - 4Bt\sin(2t) - 4C\cos(2t) - 2D\sin(2t)$
$+ 2B\cos(2t) - 2D\sin(2t) - 4Dt\cos(2t)$

$y'' + 4y = -4A\sin 2t + 4A\sin 2t - 4Bt\sin 2t + 4Bt\sin 2t - 4C\cos(2t) + 4C\cos(2t)$
$- 4Dt\cos(2t) + 4Dt\cos(2t) + 4B\cos(2t) - 4D\sin(2t) = 3\sin 2t$

$B = 0$ and $D = -3/4$.

$y = -\frac{3}{4}t\cos(2t)$? $y' = -\frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t)$

$y'' = \frac{3}{2}\sin(2t) + \frac{3}{2}\sin(2t) + 3t\cos 2t$

Then $y'' + 4y = 3\sin(2t) + 3t\cos(2t) - 3t\cos(2t) = 0.$

So $y(t) = -\frac{3}{4}t\cos(2t)$ is a solution.

Then the gen. sol. is $y(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos 2t.$

$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) - \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t)$

$y(0) = c_1$, $y'(0) = 2c_2 - 3/4$

$y(0) = 2$ so $\boxed{c_1 = 2}$

$y'(0) = -1$ so $2c_2 - 3/4 = -1$ so $2c_2 = -1/4$ so $\boxed{c_2 = -1/8}$

So $\boxed{y(t) = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3}{4}t\cos(2t)}$

3.6

(10) $y'' - 2y' + y = \frac{e^t}{1+t^2}$.

$y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$
so $r = 1$.

So $y(t) = c_1 e^t + c_2 t e^t$ solves the homogeneous eq.

Let $y(t) = u_1(t)e^t + u_2(t) t e^t$

$y'(t) = u_1(t)e^t + u_1'(t)e^t + u_2(t)e^t + u_2(t) t e^t + u_2'(t) t e^t$.

Suppose $u_1'(t)e^t + u_2'(t) t e^t = 0$.
Then $y'(t) = u_1(t)e^t + u_2(t)e^t + u_2(t) t e^t$.
$y''(t) = u_1'(t)e^t + u_1(t)e^t + u_2'(t)e^t + u_2(t)e^t + u_2'(t)te^t$
$+ u_2(t)e^t + u_2(t) t e^t$.
Since $u_1'(t)e^t + u_2'(t) t e^t$, then
$y''(t) = u_1(t)e^t + u_2'(t)e^t + 2u_2(t)e^t + u_2(t) t e^t$.

Now $y'' - 2y' + y = u_1(t)e^t + u_2'(t)e^t + 2u_2(t)e^t + u_2(t) t e^t$
$- 2u_1(t)e^t - 2u_2(t)e^t - 2u_2(t)te^t + u_1(t)e^t + u_2(t) t e^t$
$= e^t(u_1(t) + u_2'(t) + 2u_2(t) - 2u_1(t) - 2u_2(t) + u_1(t))$
$+ te^t(u_2(t) - 2u_2(t) + u_2(t))$
$= u_2'(t) e^t$.

Then $u_2'(t) = \frac{1}{1+t^2}$ so $u_2(t) = \int \frac{1}{1+t^2}\,dt$.

Let $t = \tan\theta$, then $dt = \sec^2\theta$ so $\int \frac{1}{1+t^2}\,dt = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \theta = \arctan t$.

So $u_2(t) = \arctan t + C$.

Now $u_1'(t)e^t + u_2'(t) t e^t = 0$
$u_1'(t) + u_2'(t) t = 0$
$u_1'(t) + \frac{t}{1+t^2} = 0$ so $u_1'(t) = \frac{-t}{1+t^2}$.

Then $u_1(t) = -\int \frac{t}{1+t^2}\,dt$. Let $u = 1+t^2$, $du = 2t\,dt$

so $u_1(t) = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln(1+t^2)$

Then $\boxed{y(t) = c_1 e^t + c_2 t e^t - \frac{1}{2}\ln(1+t^2)e^t + (\arctan t)te^t}$

(15) $ty'' - (1+t)y' + y = t^2 e^{2t}$, $t>0$. $y_1(t) = 1+t$, $y_2(t) = e^t$.

Let's first verify y_1 and y_2 are solutions of the homogeneous eqn.

$y_1' = 1$, $y_1'' = 0$

$ty_1'' - (1+t)y_1' + y_1 = 0 - (1+t)(1) + (1+t) = 0$.
So y_1 is a solution.

$y_2 = e^t$, $y_2' = e^t$, $y_2'' = e^t$

$ty_2'' - (1+t)y_2' + y_2 = e^t\left(t - (1+t) + 1\right) = e^t(0) = 0$.

So y_2 is a solution.

Now let's solve the nonhomogeneous DFQ.

Suppose $y = u_1(t)(1+t) + u_2(t)e^t$.
Then $y' = u_1'(t)(1+t) + u_1(t) + u_2'(t)e^t + u_2(t)e^t$.
Suppose $u_1'(t)(1+t) + u_2'(t)e^t = 0$.

Then $y' = u_1(t) + u_2(t)e^t$
$y'' = u_1'(t) + u_2'(t)e^t + u_2(t)e^t$.

$ty'' - (1+t)y' + y = tu_1'(t) + tu_2'(t)e^t + tu_2(t)e^t - u_1(t) - u_2(t)e^t$
$- tu_1(t) - tu_2(t)e^t + u_1(t) + u_1(t)t + u_2(t)e^t$
$= tu_1'(t) + tu_2'(t)e^t$

So $t u_1'(t) + t u_2'(t) e^t = t^2 e^{2t}$

and $u_1'(t)(1+t) + u_2'(t) e^t = 0$

$$u_1'(t) + u_2'(t) e^t = t e^{2t}$$
$$(1+t) u_1'(t) + u_2'(t) e^t = 0$$

After subtracting we get $-t u_1'(t) = t e^{2t}$

so $u_1'(t) = -e^{2t}$

so $\boxed{u_1(t) = -\frac{1}{2} e^{2t}}$

while $u_2'(t) e^t = -(1+t) u_1'(t) = (1+t) e^{2t}$

so $u_2'(t) = (1+t) e^t$.

$u = t \quad dv = e^t dt$
$du = dt \quad v = e^t$

So $u_2(t) = \int (1+t) e^t dt = e^t + \int t e^t dt$

$= e^t + t e^t - \int e^t dt$

$\boxed{u_2(t) = t e^t.}$

so $y(t) = -\frac{1}{2} e^{2t}(1+t) + t e^t e^t$

$= e^{2t}\left(t - \frac{1+t}{2}\right) = \boxed{\left(\frac{t-1}{2}\right) e^{2t}}$

That's a particular solution (which is what the question asks).

A general soln would be $y(t) = c_1(1+t) + c_2 e^t + \left(\frac{t-1}{2}\right) e^{2t}$.

Alternative Solution

Suppose $y = u_1(t)(1+t) + u_2(t) e^t$.

Then by theorem 3.6.1

$$Y = -y_1(t) \int_{t_0}^{t} \frac{y_2(s) g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(y_1, y_2)(s)} ds$$

$y_1 = 1+t \qquad y_2 = e^t$

$g(t) = \frac{t^2 e^{2t}}{t} = te^{2t}.$

$W(y_1,y_2)(t) = y_1 y_2' - y_1' y_2$
$= (1+t)e^t - e^t = te^t$

So $Y = -(1+t)\int \frac{(e^t)\,te^{2t}}{te^t}\,dt + e^t\int \frac{(1+t)\,te^{2t}}{te^t}\,dt$

$= -(1+t)\int e^{2t}\,dt + e^t\int (1+t)e^t\,dt$

$= (-1+t)\left(\frac{1}{2}e^{2t}\right) + e^t\left(e^t + te^t - e^t\right)$

$= \frac{-(1+t)}{2}e^{2t} + te^{2t} = e^{2t}\left(t - \frac{1+t}{2}\right)$

$= e^{2t}\left(\frac{t-1}{2}\right)$

So $\boxed{Y(t) = e^{2t}\left(\frac{t-1}{2}\right)}$

3.7

(6) $m = 100\,g$

$L = 5\,cm$

Initial velocity $10\,cm/s$ so $u'(0) = 10\,cm/s$

Start at eq. pos. so $u(0) = 0\,cm$

$g = 9.8\,\frac{m}{s^2} = 980\,\frac{cm}{s^2}$

$$mg = kL \quad \text{so} \quad k = \frac{(100g)(980\,\frac{cm}{s^2})}{5cm} = \frac{98\,\frac{kg}{s^2}}{5} = 19.6\,\frac{kg}{s^2}$$

We know $mu'' + ku = 0$

so $.1u'' + 19.6u = 0$

$u'' + 196u = 0$

$r^2 + 196 = 0$ so $r = 14i$

Then $u(t) = A\cos(14t) + B\sin(14t)$

$u(0) = 0$

so $A = 0$

$u'(t) = -14\sin(14t) + 14B\cos(14t)$

$u'(0) = 14B$ so $14B = 10$ so $B = 5/7$

Then $u(t) = \frac{5}{7}\sin(14t)$

The spring first returns to its initial position when $14t = \pi$ so $t = \frac{\pi}{14}\,s$

3.8 (9) $mg = 6\text{ lb}$ (force) so $m = \frac{6\text{ lb}}{32\,(\text{ft}/\text{s}^2)} = \frac{6}{32}$ lb. (mass unit)

$k = 1\ \text{lb}/\text{in} = 12\ \text{lb}/\text{ft}$

so $\frac{6}{32}u'' + 12u = 4\cos(7t)$.

Let's solve the homogeneous first:

$\frac{6}{32}r^2 + 12 = 0$ so $r^2 = \frac{-12\cdot 32}{6} = -64$

so $r = 8i$

so $u(t) = A\cos(8t) + B\sin(8t)$.

Now let's find a solution to the nonhomogeneous example:

Let $u(t) = C\cos(7t) + D\sin(7t)$

$u'(t) = -7C\sin(7t) + 7D\cos(7t)$

$u''(t) = -49C\cos(7t) - 49D\sin(7t)$

So $\frac{6}{32}\left(-49C\cos(7t) - 49D\sin(7t)\right) + 12C\cos(7t) + 12D\sin(7t) = 4\cos(7t)$

So $\left(\frac{-49\cdot 6}{32} + 12\right)C = 4$

and $\left(-\frac{49\cdot 6}{32} + 12\right)D = 0$ so $D = 0$.

$$C = \frac{4\cdot 32}{-49\cdot 6 + 12\cdot 32} = \frac{4\cdot 32}{6(64-49)} = \frac{2\cdot 32}{3(15)} = \frac{64}{45}.$$

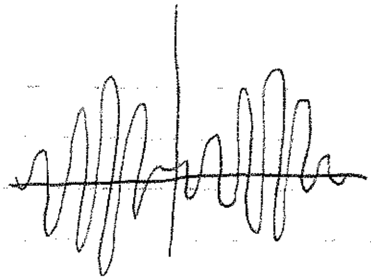
So $\boxed{u(t) = A\cos(8t) + B\sin(8t) + \frac{64}{45}\cos(7t).}$

We know $u(0) = 0$ so $A + \frac{64}{45} = 0$

$u'(0) = 0$ so $8B = 0$

so $A = -\frac{64}{45}$ and $B = 0$

Then $u(t) = \frac{64}{45}\left(\cos(7t) - \cos(8t)\right)$



(11) $mg = 8\text{ lb}$ $\quad m = \frac{8}{32}\text{ lb}$ $\quad mg = kL$, so

$L = 6\text{ in}$, $\quad L = 0.5\text{ ft}$

$\gamma = 0.25\ \frac{\text{lb}\cdot\text{s}}{\text{ft}}$ $\quad F = 4\cos(2t)$ $\quad k = \frac{mg}{L} = \frac{8}{.5} = 16.$

a) $mu'' + \gamma u' + ku = F$

$\frac{8}{32}u'' + \frac{1}{4}u' + 16u = 4\cos(2t).$

Let $u = A\cos(2t) + B\sin(2t)$.

Then $u' = -2A\sin(2t) + 2B\cos(2t)$

$u'' = -4A\cos(2t) - 4B\sin(2t)$

$4\cos(2t) = \frac{8}{32}\left(-4A\cos(2t) - 4B\sin(2t)\right) + \frac{1}{4}\left(-2A\sin(2t) + 2B\cos(2t)\right) + 16\left(A\cos(2t) + B\sin(2t)\right)$

$\cos(2t)\left(-A + \frac{1}{2}B + 16A\right) + \sin(2t)\left(-B - \frac{1}{2}A + 16B\right) = 4\cos(2t)$

$4 = 15A + \frac{1}{2}B$ so $8 = 30A + B$

$0 = 30B - A$

$240 = 900A + 30B$

$0 = -A + 30B$ so $901A = 240$ so $A = \frac{240}{901}$ and $B = \frac{A}{30} = \frac{8}{901}$

So $u(t) = \frac{240}{901}\cos(2t) + \frac{8}{901}\sin(2t)$ is the steady solution

b) Suppose m is a variable, let's maximize the amplitude.

$u = A\cos(2t) + B\sin(2t)$ implies $\cos(2t)\left(16A - 4mA + \frac{1}{2}B\right) + \sin(2t)\left(16B - 4mB - \frac{1}{2}A\right) = 0$

$(16-4m)A + \frac{1}{2}B = 4$ $\quad (32-8m)A + B = 8$ $\quad A(32-8m)^2 + (32-8m)B = 8(32-8m)$

$(16-4m)B - \frac{1}{2}A = 0$ $\quad (32-8m)B - A = 0$ $\quad -A + (32-8m)B = 0$

So $\left((32-8m)^2 + 1\right)A = 8(32-8m)$

$A = \frac{8(32-8m)}{(32-8m)^2+1}$, $B = \frac{8}{(32-8m)^2+1}$

Max amplitude when $32 = 8m$ so $m = 4$