# Homework Solutions

**4.1**

(2) $t y''' + \sin t\, y'' + 3y = \cos t$

$$y''' + \frac{\sin t}{t} y'' + \frac{3}{t} y = \frac{\cos t}{t}$$

$\frac{\sin t}{t}$, $\frac{3}{t}$ and $\frac{\cos t}{t}$ are continuous whenever $t \neq 0$.

So there are solutions on the intervals $\boxed{(-\infty, 0) \text{ and } (0, \infty).}$

(6) $y^{(6)} + \frac{x^2}{x^2-4} y''' + \frac{9}{x^2-4} = 0.$

$\frac{x^2}{x^2-4}$ and $\frac{9}{x^2-4}$ are continuous as long as $x^2 \neq 4$, i.e. $x \neq \pm 2$.

Therefore the intervals are $\boxed{(-\infty,-2), (-2,2), (2,\infty).}$

(8) $f_1 = 2t-3$, $f_2 = 2t^2+1$, $f_3 = 3t^2+t$

Suppose $k_1 f_1 + k_2 f_2 + k_3 f_3 = 0$ for all $t$.

then $(2t-3)k_1 + (2t^2+1)k_2 + (3t^2+t)k_3 = 0$

$(2k_2 + 3k_3)t^2 + (2k_1 + k_3)t + (k_2 - 3k_1) = 0.$

Then $2k_2 + 3k_3 = 0$

$2k_1 + k_3 = 0$

$k_2 - 3k_1 = 0.$

$k_3 = -\frac{2}{3} k_2$

$k_3 = -2k_1$

Let $k_3 = -2$ so $k_2 = 3$ and hence $k_1 = 1$.

Then $k_2 - 3k_1 = 0$.

So $f_1 + 3f_2 - 2f_3 = 0$

So $f_1, f_2, f_3$ are linearly dependent with linear relation $\boxed{1 f_1 + 3f_2 - 2f_3 = 0}$

$\boxed{4.2}$ (13) $2y''' - 4y'' - 2y' + 4y = 0$

$2r^3 - 4r^2 - 2r + 4 = 0$

$r^3 - 2r^2 - r + 2 = 0$

$r = 1$ is a solution since $1^3 - 2(1)^2 - 1 + 2 = 0$.

$$\begin{array}{r|l} & r^2 - r - 2 \\ \hline r-1 & r^3 - 2r^2 - r + 2 \\ & -r^3 + r^2 \\ \hline & \quad -r^2 - r + 2 \\ & \quad r^2 - r \\ \hline & \qquad -2r + 2 \\ & \qquad 2r - 2 \\ \hline & \qquad\quad 0 \end{array}$$

Factor as

$(r-1)(r^2 - r - 2)$

$= (r-1)(r-2)(r+1)$.

Then $\boxed{y(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{-t}}$

(16) $y^{(4)} - 5y'' + 4y = 0$.

$r^4 - 5r^2 + 4 = 0$. Then $x^2 - 5x + 4 = 0$ for $x = r^2$.

So $(x-4)(x-1) = 0$. So $x = 4$ and $x = 1$.

Then $r^2 = 4$ and $r^2 = 1$. So $r = \pm 2$ and $r = \pm 1$.

Then $\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{t} + c_4 e^{-t}}$

(17) $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$

$r^6 - 3r^4 + 3r^2 - 1 = 0$.

Then $x^3 - 3x^2 + 3x - 1 = 0$, where $x = r^2$.

$x^3 - 3x^2 + 3x - 1 = (x-1)^3$

So $(x-1)^3 = 0$ so $x = 1$. Then $r^2 = 1$.

$r^6 - 3r^4 + 3r^2 - 1 = (r^2 - 1)^3 = (r-1)^3 (r+1)^3$

The solutions that come from $r=1$ are $e^t$, $te^t$ and $t^2e^t$.
The solutions that come from $r=-1$ are $e^{-t}$, $te^{-t}$ and $t^2e^{-t}$.

Therefore $\boxed{y(t) = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{-t} + c_5te^{-t} + c_6t^2e^{-t}}$

(21) $y^{(8)} + 8y^{(4)} + 16y = 0$

$r^8 + 8r^4 + 16 = 0$

$(r^4+4)^2 = 0$ $\qquad r^4+4=0$ implies $r^4=-4$.

So $r^2 = \pm 2i$ $\quad$ so $\quad r = \pm\sqrt{2}\,(\sqrt{i})$

$i = e^{\frac{\pi}{2}i}$ $\quad$ so $\quad \sqrt{i} = e^{\frac{\pi}{4}i} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$

or $\quad \sqrt{i} = e^{\frac{5\pi}{4}i}$ $\quad$ since $\quad e^{\frac{10\pi}{4}i} = e^{(2\pi+\frac{1}{2}\pi)i}$

$$= \cos(2\pi+\tfrac{\pi}{2}) + i\sin(2\pi+\tfrac{\pi}{2})$$
$$= \cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2}) = i.$$

So the four roots are $\sqrt{2}\,(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$,
$\sqrt{2}\,(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}))$,
$-\sqrt{2}\,(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$
$-\sqrt{2}\,(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}))$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos\frac{5\pi}{4} = +\frac{\sqrt{2}}{2}$
$\sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

~~So~~ $\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1+i$

$\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1-i$

$-\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -1-i$

$-\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -1+i$

$e^{(1+i)t} = e^t(\cos(t) + i\sin(t))$

$e^{(1-i)t} = e^t(\cos(t) - i\sin(t))$

$e^{(-1-i)t} = e^{-t}(\cos(-t) + i\sin(-t))$
$= e^{-t}(\cos(t) - i\sin(t))$

$e^{(-1+i)t} = e^{-t}(\cos(-t) + i\sin(t))$
$= e^{-t}(\cos(t) + i\sin(t))$.

So the solutions are $e^t\cos t$, $e^t\sin t$, $e^{-t}\cos t$, $e^{-t}\sin t$, $te^t\cos t$, $te^t\sin t$, $te^{-t}\cos t$,

So $y(t) = c_1e^t\cos t + c_2e^t\sin t + c_3e^{-t}\cos t + c_4e^{-t}\sin t + c_5te^t\cos t + c_6te^t\sin t$
$+ c_7te^{-t}\cos t + c_8te^{-t}\sin t$

$\boxed{4.3}$

(3) $y''' + y'' + y' + y = e^{-t} + 4t$

Homogeneous: $y''' + y'' + y' + y = 0$, so $r^3 + r^2 + r + 1 = 0$

$1 + r + r^2 + r^3 = \frac{r^4 - 1}{r - 1}$

So $r \neq 1$ but $r^4 = 1$ so

$$r = e^{\frac{2\pi i}{4}},\ e^{\frac{4\pi i}{4}},\ e^{\frac{6\pi i}{4}}$$
$$= e^{\frac{\pi i}{2}},\ e^{\pi i},\ e^{\frac{3\pi i}{2}}$$
$$= i,\ -1,\ -i$$

So the general solution to the homogeneous equation is

$$y(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t$$

(Alternative solution to $r^3 + r^2 + r + 1 = 0$:

$$r^3 + r^2 + r + 1 = \frac{r^4 - 1}{r - 1} = \frac{(r^2-1)(r^2+1)}{r-1} = \frac{\cancel{(r-1)}(r+1)(r^2+1)}{\cancel{r-1}} = (r+1)(r^2+1)$$

so $r = -1$ and $r^2 = -1$ so $r = -1, i, -i$.)

Now let's solve the non-homogeneous eqn.

Let $Y(t) = Ate^{-t} + Bt + C$

$Y'(t) = Ae^{-t} - Ate^{-t} + B$

$Y''(t) = -2Ae^{-t} + Ate^{-t}$

$Y'''(t) = 3Ae^{-t} - Ate^{-t}$.

$$Y''' + Y'' + Y' + y = 3Ae^{-t} - Ate^{-t} - 2Ae^{-t} + Ate^{-t}$$
$$+ Ae^{-t} - Ate^{-t} + B$$
$$+ Ate^{-t} + Bt + C$$
$$= 2Ae^{-t} + Bt + (B + C) = e^{-t} + 4t$$

$\therefore$ $2A = 1$, $B = 4$, $B + C = 0$ so $A = 1/2$, $B = 4$, $C = -4$.

so $Y(t) = \frac{1}{2} te^{-t} + 4t - 4$

So $\boxed{y(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t + \frac{1}{2} te^{-t} + 4t - 4}$

(5) $y^{(4)} - 4y'' = t^2 + e^t$.

HOMOGENEOUS: $y^{(4)} - 4y'' = 0$

so $r^4 - 4r^2 = 0$

so $r^2(r^2-4) = 0$

$r = 0, \quad r = 2, \quad r = -2$.

Then $y(t) = c_1 t + c_2 + c_3 e^{2t} + c_4 e^{-2t}$

~~Let~~ $Y(t) = At^2 + Be^t$

$Y'(t) = 2At + Be^t$

$Y''(t) = 2A + Be^t$

$Y'''(t) = Be^t$

$Y^{(4)}(t) = Be^t$

$Y^{(4)} - 4Y'' = Be^t - 4(2A + Be^t) = -8A - 3Be^t$

$= t^2 + e^t$ does not work.

~~So~~ let $Y(t) = At^4 + Be^t$, $Y' = 4At^3 + Be^t$, $Y'' = 12At^2 + Be^t$,

then $Y''' = 24At + Be^t$, $Y^{(4)} = 24A + Be^t$

$Y^{(4)} - 4Y'' = Be^t - 4(12At^2 + Be^t) + 24A$

$= -3Be^t - 48At^2 + \underline{24A}$

Then $-3B = 1$ and $-48A = 1$ ↑ can't be zero.

So $B = -1/3$ and $A = -1/48$

we need another bit. Add $Ct^2$ to $Y(t)$

Then $Y^{(4)} - 4Y'' = -3Be^t - 48At + 24A - 8C$

so $Y(t) = -\frac{1}{48}t^4 - \frac{1}{3}e^t - \frac{1}{16}t^2$

Then $y(t) = c_1 t + c_2 + c_3 e^{2t} + c_4 e^{-2t} - \frac{1}{48}t^4 - \frac{1}{3}e^t - \frac{1}{16}t^2$

Then

$24A - 8C = 0$

$8C = 24A$

$C = 3A$

$= -1/16$

The solutions that come from $r=1$ are $e^t$, $te^t$ and $t^2e^t$.
The solutions that come from $r=-1$ are $e^{-t}$, $te^{-t}$ and $t^2e^{-t}$.

Therefore $\boxed{y(t) = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{-t} + c_5te^{-t} + c_6t^2e^{-t}}$

(21) $y^{(8)} + 8y^{(4)} + 16y = 0$

$r^8 + 8r^4 + 16 = 0$

$(r^4+4)^2 = 0$ $\quad$ $r^4+4=0$ implies $r^4 = -4$.

So $r^2 = \pm 2i$ $\quad$ so $\quad r = \pm\sqrt{2}\,(\sqrt{i})$

$i = e^{\frac{\pi}{2}i}$ $\quad$ so $\sqrt{i} = e^{\frac{\pi}{4}i} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$

or $\sqrt{i} = e^{\frac{5\pi}{4}i}$ $\quad$ since $e^{\frac{10\pi}{4}i} = e^{(2\pi+\frac{1}{2}\pi)i}$

$= \cos(2\pi+\frac{\pi}{2}) + i\sin(2\pi+\frac{\pi}{2})$

$= \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i.$

So the four roots are $\sqrt{2}\,(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$,

$\sqrt{2}\,(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}))$,

$-\sqrt{2}\,(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

$-\sqrt{2}\,(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}))$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{5\pi}{4} = +\frac{\sqrt{2}}{2}$

$\sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

~~So $\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1+i$~~

$\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1-i$

~~$-\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -1-i$~~

$-\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -1+i$

$e^{(1+i)t} = e^t(\cos(t) + i\sin(t))$

$e^{(1-i)t} = e^t(\cos(t) - i\sin(t))$

~~$e^{(-1-i)t} = e^{-t}(\cos(-t) + i\sin(-t))$~~

$= e^{-t}(\cos(t) - i\sin(t))$

$e^{(-1+i)t} = e^{-t}(\cos(-t) + i\sin(t))$

$= e^{-t}(\cos(t) + i\sin(t)).$

So the solutions are $e^t\cos t$, $e^t\sin t$, $e^{-t}\cos t$, $e^{-t}\sin t$, $te^t\cos t$, $te^t\sin t$, $te^{-t}\cos t$, $te^{-t}\sin t$

So $y(t) = c_1e^t\cos t + c_2e^t\sin t + c_3e^{-t}\cos t + c_4e^{-t}\sin t + c_5te^t\cos t + c_6te^t\sin t$